Lecture 10: 12.7: Cylindrical and Spherical Coordinates. Recall that in the plane it is sometimes useful to introduce polar coordinates. There are two possible natural and useful generalizations of this to space:

Cylindrical coordinates (r, θ, z) of a point $P(x, y, z)$ are obtained by using polar coordinates (r, θ) of the projection in the x-y plane and leaving z unchanged:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z \tag{12.7.1}$$

To convert from rectangular to cylindrical coordinate we use:

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z. \tag{12.7.2}$$

Note that the surfaces $r = c$ are cylinders of all points at distance c from the z-axis. The surfaces $\theta = c$ are half planes from the z-axis at an angle c with the x-z plane. The surface $z = c$ are the planes parallel to and at a distance c from the x-y plane.

Spherical coordinates (ρ, θ, ϕ) of a point $P(x, y, z)$ are obtained by first using polar coordinates of the projection in the x-y plane. We use the angle θ in the x-y plane but instead of the distance r to the z axis use the distance to the origin ρ:

$$\rho^2 = x^2 + y^2 + z^2 \tag{12.7.3}$$

Then $\rho^2 = r^2 + z^2$ and we introduce another angle ϕ to describe the hight above the x-y coordinate plane: $z = \rho\cos\phi$ and $r = \rho\sin\phi$. We obtain the coordinates:

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\cos\theta, \qquad z = \rho\cos\phi \tag{12.7.4}$$

where we must have

$$\rho \geq 0, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi\phi \tag{12.7.5}$$

Note that the surfaces $\rho = c$ are spheres of radius c.
The surfaces $\theta = c$ are half planes from the z-axis at an angle c with the x-z plane.
The surfaces $\phi = c$ are half cones at an angle c with the positive z-axis.

Ex. Write down the equations in both cylindrical and spherical coordinates.
(a) $z^2 = x^2 + y^2$, $z \geq 0$, (b) $z^2 = x^2 + y^2 + 1$, (c) $x^2 + y^2 + z^2 = 4$.

Sol. (a) $z^2 = r^2$ in cylindrical coordinates and $\phi = \pi/4$ in spherical coordinates.
(b) $z^2 = r^2 + 1$ in cylindrical coordinates and $\rho^2\cos^2\phi = \rho^2\sin^2\phi + 1$ in spherical.
(c) $z^2 + r^2 = 4$ in cylindrical coordinates and $\rho^2 = 4$ in spherical coordinates.

Ex. Write down in in cylindrical and spherical coordinates the inequalities for the solid region $\{(x, y, z); x^2 + y^2 + z^2 \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$.

Sol. In cylindrical coordinates: $\{(r, \theta, z);\ 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/2,\ 0 \leq z \leq \sqrt{1 - r^2}\}$ and in spherical coordinates it is $\{(\rho, \theta, \phi);\ 0 \leq \rho \leq 1,\ 0 \leq \theta \leq \pi/2,\ 0 \leq \phi \leq \pi/2\}$.

Review for Midterm 1, Chapter 10.-10.5,12.1-12.6:

Parametric equations for a curve: $x = f(t)$, $y = g(t)$, where $\alpha \le t \le \beta$.
Slope of tangent line $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$. Eq. of tanget line $y - y_0 = m(x - x_0)$.

Arc length: $L = \displaystyle\int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$

Area of surface of revolution around the x-axis: $A = \displaystyle\int_\alpha^\beta 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$
Polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = y/x$.

Chap. 12: Vectors If $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are two points then the vector between them is $\overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$.

Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ be vectors and θ the angle between them.
Length: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Dot product: $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$, $\quad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$.
$\mathbf{a}$ is parallel to $\mathbf{b}$ if $\mathbf{a} = c\mathbf{b}$, for some scalar c. $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if $\mathbf{a}\cdot\mathbf{b} = 0$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector $\mathbf{proj}_{\mathbf{a}}(\mathbf{b}) = \mathrm{comp}_{\mathbf{a}}(\mathbf{b})\mathbf{u}$, where $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$ is a unit vector in the direction of $\mathbf{a}$ and $\mathrm{comp}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b}\cdot\mathbf{u}$ is the component of $\mathbf{b}$ in the direction of $\mathbf{a}$.
Cross product: $\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2\,,\ a_3b_1 - a_1b_3\,,\ a_1b_2 - a_2b_1\rangle$, if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k},$$

$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$, $\quad \mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

The area of the triangle with $\mathbf{a}$ and $\mathbf{b}$ as two edges is $A = |\mathbf{a}\times\mathbf{b}|/2$.
The volume of the parallelepiped with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as three edges is $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.
$\mathbf{a}\times\mathbf{b} = 0$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$ if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ lie in the same plane.

Parametric equations of a line: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ where (x_0, y_0, z_0) is on the line and $\langle a, b, c\rangle$ is parallel to it: $\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$.

Equations of planes $ax + by + cz + d = 0$, where $\mathbf{n} = \langle a, b, c\rangle$ is a normal to the plane, i.e. it is perpendicular to it. If $P_0(x_0, y_0, z_0)$ is a point in the plane this can also be written as $\langle a, b, c\rangle\cdot\langle x - x_0, y - y_0, z - z_0\rangle = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

The distance between a point $P_1(x_1, y_1, z_1)$ and the plane is $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Examples of Quadratic surfaces: An Ellipsoid $x^2 + y^2 + 2z^2 = 1$, an Elliptic Paraboloid $z = x^2 + y^2$, a Hyperbolic Paraboloid $z = x^2 - y^2$, a Cone $z^2 = x^2 + y^2$ and a Hyperboloid $z^2 = x^2 + y^2 + 1$.