Section 13.1: Vector valued functions and space curves.

A **vector valued function** $\mathbf{r}$, is a function that to each parameter value t, in some interval I, assigns a vector $\mathbf{r}(t)$, depending on t. In components:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \qquad t \in I \tag{13.1.1}$$

Here $f(t)$, $g(t)$ and $h(t)$ are called the **component functions**.
Ex. $\mathbf{r}(t) = \langle \cos t, \sin t, t\ln|t|\rangle$, $t \neq 0$, defines a vector valued function.
The **limit** of a vector function $\mathbf{r}$ is defined by taking the limit of each component function:

$$\lim_{t\to a} \mathbf{r}(t) = \Big\langle\ \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t)\ \Big\rangle \tag{13.1.2}$$

The limit exist if the limit of each component function exist.
Ex. Find $\lim\limits_{t\to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = \langle \cos t, \sin t, t\ln|t|\rangle$.

Sol. $$\lim_{t\to 0} \mathbf{r}(t) = \Big\langle\ \lim_{t\to 0} \cos(t),\ \lim_{t\to 0} \sin(t),\ \lim_{t\to 0} t\ln|t|\ \Big\rangle = \big\langle\, 1,\ 0,\ 0\,\big\rangle.$$

A vector valued function is called **continuous** at a if $\lim\limits_{t\to a} \mathbf{r}(t)$ exist.
Thus $\mathbf{r}$ is continuous at a if and only if the component functions are.
Ex. $\mathbf{r}(t) = \langle \cos t, \sin t, t\ln|t|\rangle$ is continuous for all values of $t = a$, in particular at 0.

The set of points C such that

$$x = f(t), \qquad y = g(t), \qquad z = h(t) \tag{13.1.3}$$

for $t \in I$, is called a **space curve** in **parametric form**.
We can think $(f(t), g(t), h(t))$ as the position of a moving particle at time t.
The position vector $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ for the point $P(f(t), g(t), h(t))$ is then a vector function. Another example of a vector function is the velocity of the particle.

Ex. Sketch and describe the curve whose vector function is $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$.

Sol. The curve is a helix. Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ it follows that the curve lies on the circular cylinder $x^2 + y^2 = 1$. The z coordinate therefore spirals upward around the cylinder counter clockwise as t increases.

Ex. Find the vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the hyperbolic paraboloid $z = y^2 - x^2$.

Sol. Let $x = \cos t$, $y = \sin t$ and $z = \sin^2 t - \cos^2 t$.

13.2 Derivatives and integrals of vector functions.

The **derivative** $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in the same way as for functions

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{13.2.1}$$

if the limit exist. If $\mathbf{r}(t+h)$ is the position vector for a point Q and $\mathbf{r}(t)$ is the position vector for a point P. Then the vector $(\mathbf{r}(t+h) - \mathbf{r}(t)) = \overrightarrow{PQ}$ points in a direction that is close to the tangent line of the curve. Hence if the limit above exist we expect it to point in the direction of the tangent line. Therefore $\mathbf{r}'(t)$ is called the **tangent vector** to the curve at the point $\mathbf{r}(t)$. The unit tangent vector is given by $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$. If $\mathbf{r}(t)$ is the position vector of a moving particle then $\mathbf{r}'(t)$ is the velocity of the particle at time t.

Th. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle \tag{13.2.2}$$

Pr. We have

$$\mathbf{r}'(t) = \lim_{\varepsilon\to 0} \frac{\mathbf{r}(t+\varepsilon) - \mathbf{r}(t)}{\varepsilon} = \lim_{\varepsilon\to 0} \Big\langle \frac{f(t+\varepsilon) - f(t)}{\varepsilon}, \frac{g(t+\varepsilon) - f(t)}{\varepsilon}, \frac{h(t+\varepsilon) - f(t)}{\varepsilon} \Big\rangle$$
$$= \Big\langle \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon}, \lim_{\varepsilon\to 0} \frac{g(t+\varepsilon) - f(t)}{\varepsilon}, \lim_{\varepsilon\to 0} \frac{h(t+\varepsilon) - f(t)}{\varepsilon} \Big\rangle = \langle f'(t), g'(t), h'(t)\rangle$$

Ex. Find the derivative of the vector function $\mathbf{r}(t) = \langle t, 2t^2, t^3\rangle$. Sketch the curve and the derivative at $t = 1$. Find the equation of the tangent line at $t = 1$ and find a unit tangent vector at $t = 1$.

Sol. $\mathbf{r}'(t) = \langle 1, 4t, 3t^2\rangle$ so $\mathbf{r}'(1) = \langle 1, 4, 3\rangle$. The parametric equations of the tangent line is $\langle x, y, z\rangle = \mathbf{r}(1) + t\,\mathbf{r}(1) = \langle 1, 2, 1\rangle + t\langle 1, 4, 3\rangle$ or $x = 1 + t$, $y = 2 + 4t$ and $z = 1 + 3t$.

Integrals of vector functions. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ then we define

$$\int_\alpha^\beta \mathbf{r}(t)\,dt = \Big(\int_\alpha^\beta f(t)\,dt\Big)\,\mathbf{i} + \Big(\int_\alpha^\beta g(t)\,dt\Big)\,\mathbf{j} + \Big(\int_\alpha^\beta h(t)\,dt\Big)\,\mathbf{k} \tag{13.2.3}$$

Fundamental theorem of calculus: If $\mathbf{R}'(t) = \mathbf{r}(t)$ then

$$\mathbf{R}(\beta) - \mathbf{R}(\alpha) = \int_\alpha^\beta \mathbf{r}(t)\,dt \tag{13.2.4}$$