13.3: Arc Length and Curvature. If $\mathbf{r}$ is a vector valued function

(13.3.1) $$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}, \qquad a \le t \le b$$

then we defined the **derivative** to be

(13.3.2) $$\mathbf{r}'(t) = \lim_{\triangle t \to 0} \frac{\mathbf{r}(t+\triangle t) - \mathbf{r}(t)}{\triangle t}$$

In components this means that

(13.3.3) $$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

For a curve with vector equation (13.3.1) or equivalently parametric equations

(13.3.4) $$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad a \le t \le b$$

we define the **arc length** of the curve to be

(13.3.5) $$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\,dt$$

However, since

(13.3.6) $$|\mathbf{r}'(t)| = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$$

this can also be written:

(13.3.7) $$L = \int_a^b |\mathbf{r}'(t)|\,dt$$

It is easy to see that it is the arc length if we approximate it by a Riemann sum:

(13.3.8) $$L \sim \sum_{i=0}^{n-1} |\mathbf{r}'(t_i)| \triangle t$$

where $t_i = a + i\triangle t$, $\triangle t = (b-a)/n$ and use the definition of derivative (13.3.2):

(13.3.9) $$\mathbf{r}'(t_i) \sim \frac{\mathbf{r}(t_i + \triangle t) - \mathbf{r}(t_i)}{\triangle t}$$

we obtain

(13.3.10) $$L \sim \sum_{i=0}^{n-1} |\mathbf{r}(t_i + \triangle t) - \mathbf{r}(t_i)|$$

This is exactly the length of the polygon consisting of the line segments between the vertices $\mathbf{r}(t_i)$, $i = 0, ..., n$, which is a good approximation of the arc length.

Ex. Find the arc length of the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, when $0 \le t \le 2\pi$.

Sol. We have $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, so $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. Hence $L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,2\pi$.

The same curve can be represented by different **parametrizations**:

Ex. Find the arc length of the helix: $\mathbf{r}_2(u) = \cos u^2 \mathbf{i} + \sin u^2 \mathbf{j} + u^2 \mathbf{k}$, $0 \le u \le \sqrt{2\pi}$.

Sol. We have $\mathbf{r}_2'(u) = -2u \sin u^2\,\mathbf{i} + 2u\cos u^2\,\mathbf{j} + 2u\mathbf{k}$. Hence $|\mathbf{r}'(u)| = \sqrt{4u^2 \sin^2 u^2 + 4u^2 \cos^2 u^2 + 4u^2} = 2\sqrt{2}\,u$ and

$$L = \int_0^{\sqrt{2\pi}} |\mathbf{r}_2'(u)|\,du = \int_0^{\sqrt{2\pi}} 2\sqrt{2}u\,du = \begin{bmatrix} u^2 = t, \\ 2u\,du = dt \end{bmatrix} = \int_0^{2\pi} dt = 2\sqrt{2}\pi$$

Two different parametrizations, $\mathbf{r}(t) = \mathbf{r}_2(u)$, where $u = u(t)$, leads to the same arc length. By the chain rule $\mathbf{r}'(t) = d\mathbf{r}_2(u)/dt = \mathbf{r}_2'(u)u'(t)$ and if we change variables

$$\int |\mathbf{r}_2'(u)|\,du = \begin{bmatrix} u = u(t), \\ du = u'(t)\,dt \end{bmatrix} = \int |\mathbf{r}'(t)|\,dt.$$

We define the **arc length function** by

$$s(t) = \int_a^t |\mathbf{r}'(\tau)|\, d\tau \tag{13.3.11}$$

$s(t)$ is the length of the curve from $\mathbf{r}(a)$ to $\mathbf{r}(t)$. By the Fundamental Theorem of Calculus

$$\frac{ds(t)}{dt} = |\mathbf{r}'(t)| \tag{13.3.12}$$

It is often useful to parametrize the curve by arc length because it does not depend on any particular parametrization so it is better to use if we want to study properties of the curve itself. If $t(s)$ is the inverse of the function in (13.3.12) then

$$\mathbf{r}_2(s) = \mathbf{r}(t(s)) \tag{13.3.13}$$

is a reparametrization of the curve in terms of arc length. By the chain rule $\mathbf{r}_2'(s) = d\mathbf{r}(t(s))/ds = \mathbf{r}'(t(s))dt/ds$. Hence $|\mathbf{r}_2(s)| = |\mathbf{r}'(t)|/(ds/dt) = 1$ by (13.3.12).

Ex. Reparametrize the helix with respect to the length from $t = 0$.

Sol. If $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ then $ds/dt = |\mathbf{r}'(t)| = \sqrt{2}$ and $s(t) = \int_0^t |\mathbf{r}'(\tau)|\, d\tau = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}t$, so $t(s) = s/\sqrt{2}$. Hence the curve reparametrized in terms of arc length is $\mathbf{r}_2(s) = \mathbf{r}(t(s)) = \cos(s/\sqrt{2})\,\mathbf{i} + \sin(s/\sqrt{2})\,\mathbf{j} + (s/\sqrt{2})\,\mathbf{k}$.

Curvature. First recall that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{13.3.14}$$

is a unit vector in the direction of the tangent line to the curve. If the curve is parametrized by arc length then there is no need to divide by $|\mathbf{r}'(t)|$. Note that $\mathbf{T}(t)$ changes direction very slowly when the curve is fairly straight. The rate of change of $\mathbf{T}(t)$ per unit distance along the curve, therefore tells us something about how much the curve bends. We therefore define the **curvature** of a curve to be:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| \tag{13.3.15}$$

Note that the derivative is defined in terms of the arc length s along the curve, i.e. we measure the rate of change of $\mathbf{T}$ per unit distance traveled along the curve. It is easier to compute the curvature if it is expressed in terms of t instead of s. By the chain rule $d\mathbf{T}/dt = d\mathbf{T}/ds\, ds/dt$ and by (13.3.12) $ds/dt = |\mathbf{r}'(t)|$ so

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \tag{13.3.16}$$

Ex. Show that the curvature of a circle of radius a is $1/a$.

Sol. If $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, then $\mathbf{r}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}$ and $|\mathbf{r}'(t)| = a$. If $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)| = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ then $\mathbf{T}'(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$ and therefore $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)| = 1/a$.

There are some other formulas to calculate the curvature, e.g.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{13.3.17}$$

and in the case the curve is given as a graph in the plane, $y = f(x)$,

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}. \tag{13.3.18}$$

The **geometric interpretation** of curvature is that it is one over the radius of the circle that best approximates the curve close to a point.