Lecture 13: 13.3 Cont. Recall that, if the curve is given by $\mathbf{r}(t)$ then

$$\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)| \tag{13.3.19}$$

is the unit tangent to the curve and the curvature is given by

$$\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|. \tag{13.3.20}$$

The geometric interpretation of curvature for a curve in the x-y plane is the inverse of the radius of the circle that best approximate the curve close to a point.

The same interpretation also holds for a curve in space if we note that close to a point the curve approximately moves in a plane, the so called **osculating plane**. To see this we first define the unit **normal** to the curve:

$$\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)| \tag{13.3.21}$$

Since $\mathbf{r}'(t)$ is the velocity of the curve $\mathbf{r}''(t)$ is the acceleration and $\mathbf{T}'(t)$ is the component of the acceleration perpendicular to the tangent $\mathbf{T}(t)$. In fact, since $\mathbf{T}\cdot\mathbf{T} = |\mathbf{T}(t)|^2 = 1$, it follows that $\mathbf{T}\cdot\mathbf{T}' = 0$ and therefore $\mathbf{N}\cdot\mathbf{T} = 0$, i.e. $\mathbf{N}$ is perpendicular to the tangent line. Since $\mathbf{N}$ is the direction in which the curve accelerates it follows that the osculating plane is determined by $\mathbf{T}$ and $\mathbf{N}$.
Ex. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\kappa(t)$ for the curve $\mathbf{r}(t) = \langle t^3/3, t^2, 2t\rangle$.
Sol. We have $\mathbf{r}'(t) = \langle t^2, 2t, 2\rangle$ and $|\mathbf{r}'(t)| = \sqrt{t^4+4t^2+4} = \sqrt{(t^2+2)^2} = t^2+2$ so

$$\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)| = (t^2+2)^{-1}\langle t^2, 2t, 2\rangle. \tag{13.3.22}$$

By the product rule it follows that

$$\mathbf{T}'(t) = (t^2+2)^{-1}\langle 2t, 2, 0\rangle - (t^2+2)^{-2}2t\langle t^2, 2t, 2\rangle \tag{13.3.23}$$
$$= (t^2+2)^{-2}\Big((t^2+2)\langle 2t, 2, 0\rangle - 2t, \langle t^2, 2t, 2\rangle\Big) = (t^2+2)^{-2}\langle 4t, -2t^2+4, -4t\rangle$$

A check of this is that $\mathbf{T}\cdot\mathbf{T}'=0$. We have $|\mathbf{T}'(t)| = 2(t^2+2)^{-2}\sqrt{4t^2+(t^2-2)^2+4t^2}$ $= 2(t^2+2)^{-2}\sqrt{t^4+4t^2+4} = 2(t^2+2)^{-2}\sqrt{(t^2+2)^2} = 2(t^2+2)^{-1}$ so

$$\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)| = (t^2+2)^{-1}\langle 2t, -t^2+2, -2t\rangle \tag{13.3.24}$$

Finally

$$\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)| = 2(t^2+2)^{-2}. \tag{13.3.25}$$

Chapter 14: Partial derivatives. Partial derivatives are derivatives of functions of several variables. The slope of say a function of two variables: $z = f(x, y)$ depends on in which direction in the x-y plane we go, e.g. $f(x, y) = x - y$ increases in the positive x-direction, but decreases in the positive y direction. To describe how the function changes one gives the slope in both directions. The derivatives in the x and y directions are called the **partial derivatives** with respect to x and y.

Section 14.1 Functions of several variables. A functions f of two variables is a rule that, to each (x, y), in a some subset D of the plane, assigns a real number $f(x, y)$. The set D is the **domain** of the function and the **range** of the function is the set of all values of f, i.e. $\{f(x, y);\ (x, y) \in D\}$. We often write $z = f(x, y)$, where x and y are called independent variables and z is called the dependent variable. By the **graph** of the function f we mean the set of all points (x, y, z) such that $z = f(x, y)$ for some $(x, y) \in D$, i.e. $\{(x, y, z);\ z = f(x, y),\ (x, y) \in D\}$.

Ex. Find and sketch the domain, range and graph of $f(x, y) = \dfrac{\sqrt{x - y^2}}{x^2 + (y-1)^2}$.