Lecture 14: Section 14.1.

Ex. Find and sketch the domain, range and graph of $f(x, y) = \sqrt{16 - x^2 - y^2}$.

By the **level curve** of $f(x, y)$ we mean the set of points such that $f(x, y) = k$ for some constant k. A collection of level curves is called a **contour map**.

If the function gives the hight above sea level, then the level curve are the curves of constant elevation above sea level. If the function gives the temperature then the level curves are the curves of constant temperature, also called isothermals. The contour maps are a way to describe these functions on a paper.

Ex. Sketch the level curves $f(x, y) = k$, for $k = 0, 1, 2, 3, 4$, of $f(x, y) = \sqrt{16 - x^2 - y^2}$.

Sol. The curves are circles of radius $\sqrt{16 - k^2}$, i.e 4, 3.87, 3.46, 2.65 and 0.

A **level surface** of $f(x, y, z)$ is a set of (x, y, z) such that $f(x, y, z) = k$ for some k.

Ex. Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

Sol. The sets $f(x, y, z) = x^2 + y^2 + z^2 = k$, for $k > 0$ are spheres of radius $\sqrt{k}$.

Section 14.2 Limits and continuity. We say that $f(x, y)$ has a **limit** L as $(x, y) \to (a, b)$ if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |(x, y) - (a, b)| < \delta \qquad \Longrightarrow \qquad |f(x, y) - L| < \varepsilon \tag{14.2.1}$$

We will write this as $\lim_{(x,y)\to(a,b)} f(x, y) = L$ or $f(x, y) \to L$, as $(x, y) \to (a, b)$, or in words: $f(x, y)$ approaches L as (x, y) approaches (a, b). (14.2.1) says that $f(x, y)$ stays within ε of L if (x, y) stays within a ball of radius δ around (a, b).

If the limit exists it follows that it has to be the same along every path that approaches the point. In particular the limit has to be the same along all lines through the point, independently of the direction (although this is not sufficient.)

Ex. Show that the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ does not have a limit as $(x, y) \to (0, 0)$.

Sol. Since $f(x, 0) = x^2/x^2 = 1$ and $f(0, y) = -y^2/y^2 = -1$ the limit is different along the coordinate axis so by the above reasoning the limit does not exist.

Ex. Show that the function $f(x, y) = \dfrac{x^2 y^2}{x^2 + y^2}$ has a limit as $(x, y) \to (0, 0)$.

Sol. First, the limit has to be 0 if it exist, since $f(x, 0) = 0$ and $f(0, y) = 0$.

For any given $\varepsilon > 0$ we need to find a $\delta = \delta(\varepsilon) > 0$ such that

$$|(x, y) - (0, 0)| = \sqrt{x^2 + y^2} < \delta \qquad \Longrightarrow \qquad |f(x, y) - 0| = \frac{x^2 y^2}{x^2 + y^2} < \varepsilon. \tag{14.2.2}$$

Since $x^2 \le x^2 + y^2$ and $y^2 \le x^2 + y^2$ it follows that

$$|f(x, y) - (0, 0)| = \frac{x^2 y^2}{x^2 + y^2} \le x^2 + y^2 < \varepsilon, \quad \text{if} \quad |(x, y) - (0, 0)| = \sqrt{x^2 + y^2} < \sqrt{\varepsilon}$$

Hence (14.2.2) holds with $\delta = \sqrt{\varepsilon}$.

A function $f(x, y)$ is called **continuous** at (a, b) if $f(x, y) \to f(a, b)$ as $(x, y) \to (a, b)$. We say that $f(x, y)$ is **continuous** in D if f is continuous at every point $(a, b) \in D$.

Polynomials are continuous and rational functions, i.e. a ratios of polynomials are continuous away from points where the denominator vanishes. Products and compositions of continuous functions are continuous: If $g(x, y)$ is continuous at (a,b) and $h(z)$ is continuous at $g(a, b)$ then $f(x, y) = h(g(x, y))$ is continuous at (a,b).

Ex. Show that $f(x, y) = \dfrac{x^2 y^2}{x^2 + y^2}$, defined so $f(0, 0) = 0$, is continuous at every point.

Sol. At $(a, b) \neq (0, 0)$ this follows since the denominator is non vanishing then and at $(0, 0)$ it follows since the limit exist by the previous example.