Lecture 15: Section 14.3 Partial derivatives. are derivatives of functions of several variables. The slope of a function of two variables: $z = f(x, y)$ depends on in which direction in the x-y plane we go, the function might increases in the positive x-direction, but decreases in the positive y direction. To describe how the function changes one gives the slope in both directions. The derivatives in the x and y directions are called the **partial derivatives** with respect to x and y. The partial derivative with respect to x at a point (a, b) is the slope at $x = a$ of the curve of intersection of the surface $z = f(x, y)$ with the plane $y = b$.

In other words, the **partial derivative with respect to** x at the point (a, b), denoted by $f_x(a, b)$, is the derivative of the function $g(x) = f(x, b)$ (where $y = b$ is kept constant) at the point $x = a$:

$$f_x(a,b) = g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}. \tag{14.3.1}$$

We can now also let the point we took the derivative at vary and we get the function

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}. \tag{14.3.2}$$

called the **partial derivative with respect to** x. Similarly, we define the **partial derivative with respect to** y by

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}. \tag{14.3.3}$$

There are many different notations for the partial derivatives:

$$f_x(x,y) = \partial_x f(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x}(x,y) = f_1 = \partial_1 f. \tag{14.3.4}$$

We remark that one can calculate the derivative in any direction from just knowing the derivatives in the x and y directions. In fact, just like a function of one variable is approximated by the tangent line, a function of two variables is approximated by a tangent plane, and a plane is determined by a point and two directions.
To find $f_x(x, y)$ we think of y as constant and take the derivative with respect to x

Ex. Find $f_x(x, y)$ and $f_y(x, y)$ if $f(x, y) = y^2 - x^2$. Evaluate $f_x(1, 2)$ and $f_y(1, 2)$. Sketch the cross sections with the $y = 2$ and $x = 1$ planes and the slopes in the x and y directions of at the point $(1, 2)$.

Sol. $f_x(x, y) = -2x$ and $f_y(x, y) = 2y$ so $f_x(1, 2) = -2$ and $f_y(1, 2) = 4$.

As for functions of one variable one can also take second order partial derivatives:

$$f_{xx} = (f_x)_x, \qquad f_{xy} = (f_x)_y, \qquad f_{yx} = (f_y)_x, \qquad f_{yy} = (f_y)_y \tag{14.3.5}$$

Ex. Find f_{xx}, f_{xy}, f_{yx}, f_{yy} if $f(x, y) = \ln(x^2 + y^2)$. **Sol.** $f_x = \dfrac{2x}{x^2+y^2}$, $f_y = \dfrac{2y}{x^2+y^2}$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\Big(\frac{2x}{x^2+y^2}\Big) = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2},$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\Big(\frac{2y}{x^2+y^2}\Big) = \frac{2(x^2-y^2)}{(x^2+y^2)^2},$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\Big(\frac{2x}{x^2+y^2}\Big) = \frac{-4xy}{(x^2+y^2)^2}, \quad f_{yx} = (f_y)_x = \frac{\partial}{\partial y}\Big(\frac{2x}{x^2+y^2}\Big) = \frac{-4xy}{(x^2+y^2)^2}$$

It seems like the second partial derivatives depend on the order of differentiation. However, in the example above $f_{xy} = f_{yx}$. In fact this turns out to always be true

$$f_{xy} = f_{yx} \tag{14.3.6}$$

Partial derivatives of functions of three variables $f(x, y, z)$

(14.3.9) $$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

(14.3.10) $$f_y(x, y, z) = \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

(14.3.11) $$f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

Partial differential equations in mathematical physics
Laplace equation

(14.3.6) $$u_{xx} + u_{yy} = 0$$

describes e.g. the static electric potential outside a conductor. The wave equation:

(14.3.7) $$u_{tt} = c^2 u_{xx} = 0$$

describes water waves, gravitational waves, the sun light, motion of a guitar string. Here t stands for the time and c is the speed of the wave. The heat equation

(14.3.8) $$u_t = \kappa u_{xx}$$

describes propagation of heat.

Ex. Show that $u(x, y) = \ln(x^2 + y^2)$ is a solution of Laplace equation.

Sol. By a previous example we just calculated $u_{xx} = 2(y^2 - x^2)/(x^2 + y^2)^2$ and $u_{yy} = 2(x^2 - y^2)/(x^2 + y^2)^2$ so $u_{xx} + u_{yy} = 0$.

Ex. Show that $u(t, x) = f(ct - x) + g(ct + x)$ is a solution of the wave equation for any functions f and g.

Sol. $u_t(t, x) = cf'(ct - x) + cg'(ct + x)$, $u_{tt}(t, x) = c^2 f''(ct - x) + c^2 g''(ct - x)$ and $u_x(t, x) = -f'(ct - x) + g'(ct + x)$, $u_{xx}(t, x) = f''(ct - x) + g''(ct - x)$. Hence $u_{tt}(t, x) - c^2 u_{xx}(t, x) = ... = 0$.