Lecture 16: 15.4 Tangent planes and linear approximations.

Suppose that $f(x,y)$ has continuous first partial derivatives $f_x(x,y)$ and $f_y(x,y)$. Consider the surface S given by $z=f(x,y)$ and a point $P_0(x_0,y_0,z_0)$ on it. Let C_1 and C_2 be the intersection of the surface with the planes $y=y_0$ and $x=x_0$. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P_0. The **tangent plane** to the surface S at P_0 is the plane containing P_0, T_1 and T_2. Let us derive the equation for the tangent plane. A plane through P_0 has the form: $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ and after dividing by C and moving two terms to the other side we get

$$z-z_0=a(x-x_0)+b(y-y_0) \tag{14.4.1}$$

If (14.4.1) represents the tangent plane at the point P_0 then the intersection with the plane $y=y_0$ must be the tangent line T_1:

$$z-z_0=a(x-x_0),\qquad y=y_0 \tag{14.4.2}$$

We learned in section 14.3 that $f_x(x_0,y_0)$ is the slope of T_1 at P_0 so $a=f_x(x_0,y_0)$. Similarly $f_y(x_0,y_0)$ is the slope of T_2 so $b=f_y(x_0,y_0)$. Therefore we get the equation of the tangent plane at P_0:

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \tag{14.4.4}$$

Compare with one dimensions $y=f(x)$ and tangent line $y-y_0=f'(x_0)(x-x_0)$.

Ex. find the tangent plane to $z=f(x,y)=x^2+y^2$ at the point $(1,2,5)$.

Sol. $z-5=2(x-1)+4(y-2)$.

Since the tangent plane approximates the surface close to $(1,2)$, the linear function

$$L(x,y)=5+2(x-1)+4(y-2) \tag{14.4.6}$$

is a good approximation of the function $f(x,y)=x^2+y^2$ close to the point $(1,2,5)$:

$$f(1.1,1.9)=1.1^2+1.9^2=4.82,\qquad L(1.1,1.9)=5+2(1.1-1)+4(1.9-2)=4.8$$

The **linear approximation** of the function $f(x,y)$ for (x,y) close to (x_0,y_0) is

$$f(x,y)\sim L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

and L is the **linearization** of f at (x_0,y_0). The linear approximation in one variable is $f(x)\sim f(x_0)+f'(x_0)(x-x_0)$. Recall that we called f differentiable at a if

$$\triangle y=f(x_0+\triangle x)-f(x_0)=f'(a)+\varepsilon\triangle x,\qquad \text{where}\qquad \varepsilon\to 0,\quad \text{as}\quad \triangle x\to 0$$

We say that $z=f(x,y)$ is **differentiable at** (x_0,y_0) if

$$\triangle z=f(x_0+\triangle x,y_0+\triangle y)-f(x_0,y_0)=f_x(x_0,y_0)\triangle x+f_y(x_0,y_0)\triangle y+\varepsilon_1\triangle x+\varepsilon_2\triangle y$$

where ε_1 and $\varepsilon_2\to 0$ as $(\triangle x,\triangle y)\to(0,0)$.

Ex. Show that $f(x,y)=\sqrt{|x|\,|y|}$ is not differentiable at $(0,0)$.

Sol. $\triangle z=f(\triangle x,\triangle y)-f(0,0)=\sqrt{|\triangle x|\,|\triangle y|}$ and $f_x(0,0)=f_y(0,0)=0$ so $\triangle z-(f_x(0,0)\triangle x+f_y(0,0)\triangle y)=\sqrt{|\triangle x||\triangle y|}$ does not go to zero faster than $|\triangle x|+|\triangle y|$.

The linear approximation for a function of three variables:

$$f(x,y,z)\sim f(x_0,y_0,z_0)+f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)$$

Ex. A rectangular box is measured to have sides 60 cm, 80 cm and 120 cm with an error of $\pm 1cm$ in each measurement. Find and approximation of the total volume and estimate the maximum error by the linear approximation.

Sol. $V=f(x,y,z)=xyz,\ f_x(x,y,z)=yz,\ f_y(x,y,z)=xz,\ f_z(x,y,z)=xy$

$$f(x,y,z)\sim 0.6\cdot 0.8\cdot 1.2+0.8\cdot 1.2(x-0.6)+0.6\cdot 1.2(y-0.8)+0.6\cdot 0.8(z-1.2)$$

$$=0.576+\pm(0.96+0.72+0.48)\cdot 0.01=0.576\pm 0.0216$$