

**Lecture 17: Section 14.4 cont.: Differentials.** Let  $x$  be a variable and let  $y = f(x)$  be a dependent variable. Let  $dx$  be another variable called the differential of  $x$ . Then the differential of the dependent variable  $dy$ , is defined by

$$(14.4.5) \quad dy = f'(x) dx = \frac{dy}{dx} dx$$

For fixed  $x$ ,  $dy$  is defined to be the linear function of  $dx$  so that  $(x + dx, y + dy)$  is on the tangent line to the curve  $y = f(x)$  at  $x$ . Hence if  $x$  changes an amount  $dx = \Delta x$  then the change of  $y$ :  $\Delta y = f(x + \Delta x) - f(x)$  is approximately  $dy$ .

Let  $x$  and  $y$  be a variable and let  $z = f(x, y)$  be a dependent variable. Let  $dx$  and  $dy$  be other variable called the differentials of  $x$  and  $y$ . Then the differential of the dependent variable  $dz$ , is defined by

$$(14.4.5) \quad dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

For fixed  $(x, y)$ ,  $dz$  is defined to be the linear function of  $(dx, dy)$  so that  $(x + dx, y + dy, z + dz)$  is on the tangent plane to the surface  $z = f(x, y)$  at  $(x, y)$ . Hence if  $x$  and  $y$  changes amounts  $dx = \Delta x$  and  $dy = \Delta y$  the change of  $z$ :  $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$  is approximately  $dz$ .

**Ex.** Find the differential of  $z = f(x, y) = x^2 + y^2$  at  $(1, 2)$ . If  $x$  changes from 1 to 0.9 and  $y$  from 2 to 1.9 compare the differential  $dz$  and the change  $\Delta z$ .

**Section 14.5: The chain rule.** If  $y = f(g(t))$ , i.e  $y = f(x)$  and  $x = g(t)$ , then

$$(14.5.1) \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

**Th. chain rule case 1** If  $z = f(x, y)$  is differentiable, where  $x = g(t)$  and  $y = h(t)$  are also differentiable then

$$(14.5.2) \quad \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

**Pf.** A change of  $\Delta t$  in  $t$  produces a change  $\Delta x$  in  $x$   $\Delta y$  in  $y$ . These in turn produce a change  $\Delta z$  in  $z$  which by the differentiability is

$$(14.5.3) \quad \Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where  $\varepsilon_1$  and  $\varepsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ . Dividing by  $\Delta t$ :

$$(14.5.4) \quad \frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

When  $\Delta t \rightarrow 0$  then  $(\Delta x, \Delta y) \rightarrow 0$  and we get (14.5.2) since  $\Delta x/\Delta t \rightarrow dx/dt$  and  $\Delta y/\Delta t \rightarrow dy/dt$ .

**Ex.** If  $z = x^2 + y^2$ ,  $x = \cos t$  and  $y = \sin t$  find  $dz/dt$ .

**Sol. 1** 
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2x(-\sin t) + 2y \cos t = 2 \cos t (-\sin t) + 2 \sin t \cos t = 0$$

**Sol. 2**  $z = x^2 + y^2 = \cos^2 t + \sin^2 t = 1, \quad \frac{dz}{dt} = 0$

**Th. chain rule case 2** If  $z = f(x, y)$  is differentiable, where  $x = g(s, t)$  and  $y = h(s, t)$  are also differentiable then

$$(14.5.5) \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$(14.5.6) \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

**Pr.** This reduces to the previous case by thinking of  $s$  or  $t$  as fixed since that is how the partial derivatives are defined.

**Ex.** If  $z = x^2 + y^2$ ,  $x = e^s \cos t$  and  $y = e^s \sin t$  find  $\partial z/\partial s$  and  $\partial z/\partial t$ . **Sol. 1:**

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 2x(e^s \cos t) + 2y e^s \sin t = 2e^s \cos t (e^s \cos t) + 2e^s \sin t e^s \sin t = 2e^{2s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = 2x(-e^s \sin t) + 2y e^s \cos t = 2e^s \cos t (-e^s \sin t) + 2e^s \sin t e^s \cos t = 0$$

**Sol. 2**  $z = x^2 + y^2 = e^{2s} \cos^2 t + e^{2s} \sin^2 t = e^{2s}, \quad \frac{\partial z}{\partial s} = 2e^{2s} \quad \frac{\partial z}{\partial t} = 0$

**Implicit differentiation.** If  $y = f(x)$  is given implicitly as a function of  $x$  by  $F(x, y) = 0$  then since  $F(x, y(x))$  is constant, we can think of  $y$  as a function of  $x$  and  $x$  is the identity function of  $x$  so using the chain rule:

$$(14.5.7) \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

so

$$(14.5.8) \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

**Ex.** Find  $dy/dx$  if  $y = y(x)$  is given by  $F(x, y) = y^3 x^2 - x^4 + 1 = 0$ .

**Sol.**

$$3y^2 \frac{dy}{dx} x^2 + y^3 2x - 4x^3 = 0$$

so

$$\frac{dy}{dx} = \frac{4x^3 - y^3 2x}{3y^2 x^2}$$