

## Lecture 19: 14.6 Cont..

The **gradient** of  $f(x, y)$  is the vector  $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ .

The **directional derivative** of  $f(x, y)$  in the direction of a unit vector  $\mathbf{u}$  is  $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ . The directional derivative  $D_{\mathbf{u}}$  gives the rate of change of  $f$  in the direction of  $\mathbf{u}$ . The direction of the gradient is the direction in which  $f$  increases the most and the rate of change in this direction is given by  $|\nabla f|$ . If  $f(x, y) = k$  is level curve and  $(a, b)$  a point on it then  $\nabla f(a, b)$  is perpendicular to the level curve. If you want to reach the top of a hill as fast as possible you follow the **curve of steepest ascent** since it is likely to take you there as fast as possible.

**Ex.** Find the gradient of  $f(x, y) = 1 - x^2 - y^2$  at the point  $(1, 2)$ . Find the directional derivative in the direction of  $\mathbf{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$  at the point  $(1, 2)$ . In which direction does  $f$  increase the fastest at the point  $(1, 2)$  and what is the rate of change in this direction? Say that you start at the point  $(1, 2)$  and want to reach the maximum as fast as possible. In which direction would you go?

### 14.7: Maximum and minimum values.

**Def.** A function  $f(x, y)$  has a **local maximum** at  $(a, b)$  if  $f(x, y) \leq f(a, b)$  when  $(x, y)$  is near  $(a, b)$ . The number  $f(a, b)$  is then called the **local maximum value**. Similarly,  $f(x, y)$  has a **local minimum** at  $(a, b)$  if  $f(x, y) \geq f(a, b)$  when  $(x, y)$  is near  $(a, b)$ , and then  $f(a, b)$  is then called the **local minimum value**. If the inequalities hold for all  $(x, y)$  in the domain of  $f$  then it is also called an **absolute maximum** or **global maximum** respectively **absolute minimum** or **global minimum**. By an **extreme value** we mean a maximum or minimum.

**Th.** If  $f(x, y)$  has as a maximum or minimum at  $(a, b)$  then  $f_x(a, b) = f_y(a, b) = 0$ .

**Pf.** If  $f$  has a local max at  $(a, b)$  then  $g(x) = f(x, b)$  has a local max at  $a$ , so  $g'(a) = 0$  and hence  $f_x(a, b) = g'(a) = 0$ . Similarly  $h(y) = f(a, y)$  satisfy  $f_y(a, b) = h'(b) = 0$ .

The conclusion in the theorem can also be stated  $\nabla f(a, b) = \mathbf{0}$ . The geometric interpretation is that the tangent plane to the surface  $z = f(x, y)$  at  $(a, b)$  is horizontal.

**Def.** A point  $(a, b)$  is called a **critical point** of  $f(x, y)$  if  $f_x(a, b) = f_y(a, b) = 0$ .

By the above theorem a local maximum or local minimum has to be a critical point. However, not all critical points are local maximum or minimum. For functions of one variable take e.g.  $f(x) = x^3$  at  $x = 0$ .

How do we know if a critical point is an extreme value? For functions of one variable, if  $f'(a) = 0$  and  $f''(a) > 0$  then it is a min and if  $f''(a) < 0$  then it is a max and if  $f''(a) = 0$  then it could be either.

**Th. (The second derivative test)** Suppose that  $(a, b)$  is a critical point and

$$D = f_{xx}(a, y)f_{yy}(a, b) - f_{xy}(a, b)^2.$$

If  $D > 0$  and  $f_{xx}(a, y) > 0$  then  $f(a, b)$  is a local minimum.

If  $D > 0$  and  $f_{xx}(a, y) < 0$  then  $f(a, b)$  is a local maximum.

If  $D < 0$  then  $f(a, b)$  is not a local minimum or maximum.

If  $D = 0$  then the test is inconclusive.

The case  $D < 0$  is called a saddle point. To remember the formula:

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

**Ex.** Find the critical points of  $f(x, y) = x^2 + y^2$ . Are they extreme values?

**Ex.** Find the critical points of  $f(x, y) = y^2 - x^2$ . Are they extreme values?