Lecture 20: 14.7 Cont..

Th. If $f(x, y)$ has as a local max or min at (a, b) then (a, b) is a critical point, i.e. $f_x(a, b) = f_y(a, b) = 0$.

Th.(Second derivative test.) See previous section.

Th.(Extreme value theorem) If f is continuous on a closed and bounded set D then f attains an absolute maximum and an absolute minimum in D.

To find the max and min in a closed and bounded domain:

1) Find the critical points in the domain.

2) Find the extreme values on the boundary.

Note that we do not need to use the second derivative test since by the extreme value theorem we know that there is a max and a min and any max or min either has to be a point on the boundary or a critical point in the interior.

Note also that the extreme value theorem holds in any number of dimensions. In particular in one dimension. If we want to find the maximum of $f(x)$ over the interval $I = [a, b] = \{x;\, a \leq x \leq b\}$, then we first find all the critical points $f'(c_i) = 0$, $i = 1, ..., N$ and we check the value of f on these points and the boundary points a and b in order to find the largest and smallest value.

Ex. Find the max and min of $f(x, y) = x^2 + 2y^2$ over (a) $D = \{(x, y);\, |x| \leq 1,\, |y| \leq 1\}$ and (b) $D = \{(x, y);\, x^2 + y^2 \leq 1\}$.

Sol. Critical points $f_x = 2x = 0$ and $f_y = 4y = 0$ is $(x, y) = (0, 0)$ and $f(0, 0) = 0$. (a) Extreme value on the boundary. Divide boundary into the 4 parts. (1) $x=1$ and $-1 \leq y \leq 1$: If $g(y) = f(1, y) = 1+2y^2$ then $g'(y) = 4y = 0$ if $y = 0$ and $g(0) = f(1, 0) = 1$. Endpoints of the interval: $g(1) = f(1, 1) = f(1, -1) = g(-1) = 3$. (2) $y = 1$ and $-1 \leq x \leq 1$. If $h(x) = f(x, 1) = x^2+1$ then $h'(x) = 2x = 0$ if $x = 0$ and $g(0) = f(0, 1) = 2$. Endpoints of the interval: $h(1) = f(1, 1) = f(-1, 1) = h(-1) = 3$. The other two parts of the boundary are the same so max is $f(\pm 1, \pm 1) = 3$ and min is $f(0, 0) = 0$.
(b) Ext em value on the boundary. Sol. 1: Parametrize boundary $(x, y) = (\cos t, \sin t)$. $g(t) = f(\cos t, \sin t) = \cos^2 t + 2\sin^2 t$. $g'(t) = -2\cos t \sin t + 4 \sin t \cos t = 2 \sin t \cos t = 0$ if $t = 0, \pi/2, \pi, 3\pi/2$. $g(0) = f(1, 0) = 1$, $g(\pi/2) = f(0, 1) = 2$, $g(\pi) = f(-1, 0) = 1$ and $g(3\pi/2) = f(0, -1) = 2$ so max is $f(0, \pm 1) = 2$ and min is $f(0, 0) = 0$. Sol2: Solve for y in $x^2 + y^2 = 1$ gives $y = \pm\sqrt{1 - x^2}$, $-1 \leq x \leq 1$. Substituting into $f(x, y)$ gives $h(x) = f(x, \pm\sqrt{1 - x^2}) = 2 - x^2$ and we want to maximize over $-1 \leq x \leq 1$. $h'(x) = -2x = 0$ if $x = 0$ and $h(0) = f(0, \pm 1) = 2$. Endpoints $h(\pm 1) = f(\pm 1, 0) = 1$.

14.8: Lagrange multipliers. In the previous section we found the extreme of $f(x, y) = x^2 + 2y^2$ subject to the constraint that (x, y) was on the circle $x^2 + y^2 = 1$. We solved this problem by reducing it to a problem in one dimension less by using the constraint to solve for one of the variables in terms of the other. However, there is another general geometric method called Lagrange method.

Say that we want to maximize $f(x, y)$ subject to the constraint that $g(x, y) = 0$. Let C be the curve $g(x, y) = 0$. To maximize f on C is to find the level curve $f(x, y) = k$ with largest k that intersects C. Say that k_0 is the largest value and let C_0 by the level curve $f(x, y) = k_0$. It is geometrically clear that such a level curve C_0 has to be tangential to the curve C at the point $P_0(x_0, y_0)$ with maximum value $f(x_0, y_0) = k_0$ where they intersect. In fact, assume that they are not tangential, then since $f(x, y) > k_0$ on one side of C_0 it follows that C can not go in that region. Since at the point P_0 the level curves C and C_0 are tangential it follows that the gradients ∇f and ∇g have to be parallel as well:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{14.8.1}$$

Another way to see this is to parametrize the curve C: $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Then $h(t) = f(x(t), y(t))$ has a maximum at $t = t_0$ where $\mathbf{r}(t_0) = (x_0, y_0)$. Hence $h'(t_0) = \nabla f \cdot \mathbf{r}'(t_0) = 0$, i.e. if the gradient is orthogonal to the tangent line of C, but we already know that the gradient ∇g is orthogonal to C from section 14.6. Also the constraint has to be satisfied

$$g(x_0, y_0) = 0 \tag{14.8.2}$$

Lagrange method is to find the solution of (14.8.1) and (14.8.2).
Each point satisfying these two equations has to to be a local extreme value.
Ex. Find the max of $f(x, y) = x^2 + 2y^2$ subject to the constraint $g(x, y) = x^2 + y^2 = 1$.
Sol. (14.8.1) become $\langle 2x, 4y\rangle = \lambda\langle 2x, 2y\rangle$ or $2x = \lambda 2x$ and $4y = \lambda 2y$. First we must have $\lambda \neq 0$ since $(0, 0)$ is not on the boundary. Then if $x \neq 0$ it follows that $\lambda = 1$ and hence $y = 0$ and if $y \neq 0$ it follows that $\lambda = 1/2$ and $x = 0$ so the only possibilities are $(x, 0)$ or $(0, y)$ Putting this into the constraint gives $(\pm 1, 0)$ or $(0, \pm 1)$.