Lecture 21: 14.8: Cont.. Recall that to find the max/min we first find the critical points $f_x(a,b) = f_y(a,b) = 0$ and then determine if they are max/min by using e.g. the second derivative test.

Ex. Find the minimum distance from the point $(1,2,2)$ to the plane $x+y+z=4$.

Sol. We want to minimize $F(x,y,z) = d(x,y,z)^2 = (x-1)^2+(y-2)^2+(z-2)^2$ over all (x,y,z) on the plane. Since on the plane $z = 4-x-y$ we can substitute for z and instead minimize $f(x,y) = F(x,y,4-x-y) = (x-1)^2+(y-2)^2+(x+y-2)^2$. Then $f_x = 2(x-1)+2(x+y-2) = 0$ and $f_y = 2(y-2)+2(x+y-2) = 0$ is equivalent to $4x+2y-6 = 0$ and $2x+4y-8 = 0$ which gives $(x,y) = (2/3, 5/3)$ and $f(2/3, 5/3) = 1/3$. By the second derivative test: $f_{xx}f_{yy} - f_{xy}^2 = 4\cdot 4 - 2^2 = 12 > 0$ and $f_{xx} > 0$ so it is a local min. That this indeed is the absolute minimum follows since there has to be a point on the plane with smallest distance.

Lagrange multipliers. To find the max and min of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ we find all values of (x,y,z) and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z), \qquad \text{and} \qquad g(x,y,z) = k \tag{14.8.3}$$

Evaluating $f(x,y,z)$ at all the resulting points gives the max and min.

That (14.8.3) has to hold on a max can be argued as follows. Suppose that $f(x_0,y_0,z_0) = k_0$ is the maximum value and let $S_0 = \{(x,y,z);\ f(x,y,z) = k_0\}$ and $S = \{(x,y,z);\ g(x,y,z) = k\}$. Since the gradient is normal to the tangent planes the statement (14.8.3) is equivalent to saying that the tangent planes to the two surfaces has to be parallel at (x_0,y_0,z_0). However, this is geometrically clear since the surface S has to lie completely within the region $D_0 = \{(x,y,z);\ f(x,y,z) \le k_0\}$

Ex. Find the min of $F(x,y,z) = d(x,y,z)^2 = (x-1)^2+(y-2)^2+(z-2)^2$ subject to the constraint $g(x,y,z) = x+y+z = 4$.

Sol. $2(x-1) = \lambda 1$, $2(y-2) = \lambda 1$, $2(z-2) = \lambda 1$ so $x = 1+\lambda/2$, $y = 2+\lambda/2$, o $z = 2+\lambda/2$ Plug into the constraint gives $g(1+\lambda/2, 2+\lambda/2, 2+\lambda/2) = 5+3\lambda/2 = 4$ so $\lambda = -2/3$, which gives $(x,y,z) = (2/3, 5/3, 5/3)$ and $F(2/3,5/3,5/3) = 1/3$.

To find the max and min of $f(x,y,z)$ subject to the constraints $g(x,y,z) = k$ and $h(x,y,z) = c$ we find all values of (x,y,z), λ and μ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z), \quad \text{and} \quad g(x,y,z) = k, \quad h(x,y,z) = c \tag{14.8.4}$$

Evaluating $f(x,y,z)$ at all the resulting points gives the max and min.

Ex. Find the maximum volume of a rectangular box with side lengths x, y and z whose total area is 1500 cm^2 and the total length of the edges is 200 cm.

Sol. Maximize $V(x,y,z) = xyz$ subject to the constraint $A(x,y,z) = 2(xy+yz+zx) = 1500$ and $L(x,y,z) = 4(x+y+z) = 200$, Find (x,y,z), λ and μ such that

$$\nabla V(x,y,z) = \lambda \nabla A(x,y,z) + \mu \nabla L(x,y,z), \quad A(x,y,z) = 1500, \quad \text{and} \quad L(x,y,z) = 200$$

Review:. 13. Vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.

Derivative: $\mathbf{r}'(t) = \lim_{h\to 0} \dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle f'(t), g'(t), h'(t)\rangle$, is **tangent** to the curve $\mathbf{r}(t)$

Arclength: $L = \int_a^b |\mathbf{r}'(t)|\, dt$, where $|\mathbf{r}'(t)| = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$

14. Functions of several variables $f(x,y)$ and $F(x,y,z)$.
Level curves $f(x,y) = k$ and **level surfaces** $F(x,y,z) = k$.

Partial derivatives:

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h},$$
$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Gradient: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$, $\nabla F(x,y,z) = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\rangle$

Chain Rule case 1: $\dfrac{d}{dt} F(\mathbf{r}(t)) = F_x \dfrac{dx}{dt} + F_y \dfrac{dy}{dt} = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ $\mathbf{r}(t) = \langle f(t), g(t)\rangle$

Geometrically: The gradient is orthogonal to the tangent line of the level curves.
Directional derivative in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$

$$D_{\mathbf{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h} = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y)\cdot \mathbf{u}$$

Max rate of change is $|\nabla f|$ which occurs in the direction of ∇f.
Chain Rule case 2: If $z = f(x,y)$ where $x = g(s,t)$ and $y = h(s,t)$ then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tangent plane: The tangent plane to the surface $z = f(x,y)$ at a point (x_0, y_0, z_0) :

(14.8.3) $z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$, $z_0 = f(x_0, y_0)$

The tangent plane to a level surface $F(x,y,z) = k$ at a point (x_0, y_0, z_0) is

(14.8.4) $F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0$

Geometrically: The gradient is orthogonal to the tangent plane of the level surface.
Differentials: If $z = f(x,y)$ and (dx, dy) are variables then the differential of f is

$$dz = f_x(x,y)\, dx + f_y(x,y)\, dy$$

Linear approximation: With $dx = \triangle x$ and $dy = \triangle y$ we have

$$\triangle z = f(x + \triangle x, y + \triangle y) - f(x,y) \sim dz = f_x(x,y)\, \triangle x + f_y(x,y)\, \triangle y$$

Max-min of $f(x,y)$ is a **critical point:** $f_x(x,y) = f_y(x,y) = 0$.
Second derivative test If (x,y) is a critical point and $D = f_{xx} f_{yy} - f_{xy}^2$.
Then it is a local min if $D > 0$ and $f_{xx} > 0$, a local max if $D > 0$ and $f_{xx} < 0$ and saddle point if $D < 0$, i.e. neither max nor min.

Lagrange multipliers. To find the max and min of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ we find all values of (x,y,z) and λ such that

(14.8.5) $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, and $g(x,y,z) = k$

Evaluating $f(x,y,z)$ at all the resulting points gives the max and min.