Lectures 23: 15.2. Double integral: over a rectangle $R = \{(x, y);\, a \le x \le b,\, c \le y \le d\}$

$$\iint_R f(x, y)\, dA$$

If f is positive then this is the volume of the solid S above the rectangle R in the x-y plane and below the surface $z = f(x, y)$. **Iterated integrals**:

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \Big[\int_c^d f(x, y)\, dy\Big]\, dx$$

Th.(Fubini)

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy \tag{15.2.3}$$

Pf Intuitively, we can interpret $A(x) = \int_c^d f(x, y)\, dy$ as the area of the cross section of the solid S with the plane with constant x coordinate. Then we can think of the volume of S as $\int_a^b A(x)\, dx$.

Ex. Find the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$.

Sol.

$$\iint_R 16 - x^2 - 2y^2\, dA = \int_0^2 \int_0^2 16 - x^2 - 2y^2\, dy\, dx = \int_0^2 16y - x^2 y - 2y^3/3\Big|_0^2\, dy$$

$$= \int_0^2 32 - 2x^2 - 16/3\, dx = 32x - 2x^3/3 - 16x/3\Big|_0^2 = 64 - 16/3 - 32/3 = 48$$

Ex. Find $\int_0^2 \int_0^\pi y \cos(xy)\, dy\, dx$.

Sol.

$$\int_0^2 \int_0^\pi y \cos(xy)\, dy\, dx = \int_0^\pi \int_0^2 y \cos(xy)\, dx\, dy = \int_0^\pi \sin(xy)\Big|_{y=0}^2\, dx$$

$$= \int_0^\pi \sin(2x)\, dx = \cos(2x)\Big|_{x=0}^\pi = 0$$

Useful fact:

$$\int_a^b \int_c^d g(x)h(y)\, dydx = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

15.3 Double integrals over more general regions. Theoretically, if we want to define the double integral over a more general bounded region D we find a sufficiently large rectangle R containing D; $D \subset R$. We define

(15.3.1)

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA, \quad \text{where} \quad F(x, y) = \begin{cases} f(x, y), & (x, y) \in D \\ 0, & (x, y) \notin D \end{cases}$$

Region of type I:

$$D = \{(x, y);\, a \le x \le b,\, g_1(x) \le y \le g_2(x)\} \tag{15.3.2}$$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \tag{15.3.3}$$

In fact using (15.3.1) and Fubini

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \tag{15.3.4}$$

since $F(x,y) = 0$ when $y \le g_1(x)$ or $y \ge g_2(x)$ and $F(x,y) = f(x,y)$, when $g_1(x) \le y \le g_2(x)$.

Region of type I:

$$D = \{(x,y);\, c \le y \le d,\, h_1(y) \le x \le h_2(y)\} \tag{15.3.5}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \tag{15.3.6}$$

Ex. Evaluate $\iint_D x\cos y\,dA$, where $D = \{(x,y);\, 0 \le x \le 1,\, 0 \le y \le x^2\}$.

$$\iint_D x\cos y\,dA = \int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 x\sin y\Big|_{y=0}^{x^2} dx = \int_0^1 x\sin(x^2)\,dx$$
$$= \cos(x^2)/2\Big|_0^1 = \frac{\cos 1 - 1}{2}$$

Ex. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA, \quad D = \{(x,y);\, 0 \le y \le 1,\, 3y \le x \le 3\}$$

Here

$$D = \{(x,y);\, 0 \le y \le 1,\, 3y \le x \le 3\} = \{(x,y);\, 0 \le x \le 3,\, 0 \le y \le x/3\}$$

so

$$\iint_D e^{x^2}\,dA = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx == \int_0^3 e^{x^2} y\Big|_{y=0}^{x/3} dx = \int_0^3 e^{x^2} x/3\,dx = \frac{e^{x^2}}{6}\Big|_{x=0}^3 = \frac{e^9 - 1}{6}$$

If $f(x,y) \le g(x,y)$ then $\iint_D f(x,y)\,dA \le \iint_D g(x,y)\,dA$. Furthermore, we can us double integrals to calculate areas:

$$\iint_D 1dA = \text{Area}(D).$$

This is because the integral is the volume above D and below 1 which is $\text{Area}(D)\cdot 1$. If $m \le f \le M$ then it follows that $m\text{Area}(D) \le \iint_D f\,dA \le M\text{Area}(D)$ which can be used to estimate the integral.