Lecture 24: Section 15.4: Double Integrals in Polar Coordinates. Suppose that we want to evaluate a double integral over a region R that can be more easily described in terms of polar coordinates than in rectangular coordinates, e.g. the unit disc
$D = \{(x, y);\ x^2 + y^2 \leq 1,\ \} = \{(r, \theta);\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$ or more generally a polar rectangle $R = \{(r, \theta);\ a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$. (Recall that $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$.) If f is continuous and positive then the double integral is the the volume V of the solid in x-y-z space above the polar rectangle R and below the graph of $z = f(x.y)$:

$$V = \iint_R f(x, y)\, dA$$

The surface can also be expressed in terms of polar coordinate $z = f(x, y) = f(r\cos\theta, r\sin\theta) = g(r, \theta)$ and we are going to calculate the volume V in a new way using polar coordinates. In order to compute the volume we devide $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\triangle r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\triangle\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays from the origin $\theta = \theta_j$ divide the polar rectangle R into small polar subrectangles $R_{ij} = \{(r, \theta);\ r_{i-1} \leq r \leq r_i,\ \theta_{j-1} \leq \theta \leq \theta_j\}$. The center of R_{ij} is $r^*_{ij} = (r_{i-1} + r_i)/2$ and $\theta^*_{ij} = (\theta_{j-1} + \theta_j)/2$. The area of a circle sector of angle θ and radius r is $\theta r^2/2$. (In fact if $\theta = 2\pi$ this is the area of the disc and in general the proportion of the disc covered is $\theta/(2\pi)$.) The area of R_{ij} is

$$\triangle A_{ij} = \theta r_i^2 \triangle/2 - \theta r_{i-1}^2 \triangle/2 = \theta(r_i + r_{i-1})(r_i - r_{i-1})/2 = r_i^* \triangle\theta\triangle r$$

Thus geometrically, we get an approximation for the volume as

$$V \sim \sum_{i=1}^{m}\sum_{j=1}^{n} f(r^*_{ij}\cos\theta^*_{ij}, r^*_{ij}\sin\theta^*_{ij}) r_i^* \triangle\theta\triangle r$$

On the other hand, this is also a Riemann sum for the double integral of $f(r\cos\theta, r\sin\theta)\, r$ in polar coordinates:

$$\iint_R f(r\cos\theta, r\sin\theta)\, r dr\, d\theta$$

Hence we deduce that

$$\iint_R f(x, y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r dr\, d\theta$$

Furthermore, we can convert the double integral in polar coordinates to an iterated integral in polar coordinates:

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r dr\, d\theta$$

Ex. Find the volume of the solid bounded by the plane $z = 0$ and the surface $z = \sqrt{1 - x^2 - y^2}$.
Sol.

$$V = \iint_D \sqrt{1 - x^2 - y^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2}\, r dr\, d\theta$$

$$= \int_0^{2\pi} -(1 - r^2)^{3/2}/3\big|_0^1\, d\theta = \int_0^{2\pi} d\theta/3 = 2\pi/3$$

We remark that converting it to an iterated integral in rectangular coordinates would give

$$V = \int_0^1 \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dy\, dx$$

which is not so easy to calculate.

The integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-x^2}\, dx$ plays an important role in probability. Even though we can not find an explicit form for the antiderivative of e^{-x^2} this integral can actually be calculated using a clever trick. First we note that the integral over the whole plane

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2}\, dA = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2}\, dx\, dy = \int_{-\infty}^{+\infty} e^{-x^2}\, dx \int_{-\infty}^{+\infty} e^{-y^2}\, dy$$

is the square of the integral that we want to calculate. On the other hand expressing the integral in polar coordinates:

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2}\, dA = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r dr\, d\theta = \int_0^{2\pi} -e^{-r^2}/2\Big|_0^{\infty} d\theta = \pi$$

Hence

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$$