Lecture 25: 15.6 Surface Area. In section 8.2 and in section 10.3 we calculated the area of a surface of revolution. Here we compute the area for a more general surface S of the form $z = f(x, y)$. For simplicity let us assume that $f \geq 0$ and the domain D of f is a rectangle R. Recall that we defined the length of the arc of a curve by approximating it with a polygonal curve. The corresponding for a surface would be to approximate it by triangles, but surprisingly this does not work.

Instead we divide the rectangle into smaller rectangles R_{ij} for $i = 1, ..., m$ and $j = 1, ..., n$, with side lengths $\triangle x$ and $\triangle y$ and corners (x_i, y_j), $(x_i + \triangle x, y_j)$, $(x_i, y_j + \triangle y)$ and $(x_i + \triangle x, y_j + \triangle y)$. In each rectangle we approximate the area of the surface $z = f(x, y)$ over R_{ij} by the area of the tangent plane to the surface over R_{ij}, which we call $\triangle T_{ij}$. As we make a partition into smaller rectangles we therefore define:

$$\text{(15.6.1)} \qquad \text{Area}(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \triangle T_{ij}$$

It remains to calculate the area of the piece of the tangent plane $\triangle T_{ij}$. The equation of the tangent plane is

$$\text{(15.6.2)} \quad z - z_0 = f_x(x_i, y_j)(x - x_i) + f_y(x_i, y_j)(y - y_j), \qquad \text{where} \quad z_0 = f(x_i, y_j)$$

Let $\mathbf{a}$ be the vector that starts at (x_i, y_j, z_0) and lie along the side of the parallelogram with area T_{ij} and ends at the point $(x_i + \triangle x, y_j, z_0 + \triangle z)$ where $z_0 + \triangle z$ is the point on the tangent plane (15.6.3) evaluated from taking $x = x_i + \triangle x$ and $y = y_j$, i.e. $\triangle z = f_x(x_i, y_j)\triangle x$ and $\mathbf{a} = \triangle x\, \mathbf{i} + f_x(x_i, y_j)\triangle x\, \mathbf{k}$. Similarly, let $\mathbf{b}$ be the vector that starts at (x_i, y_j, z_0) and goes to the point on the tangent plane with $x - y$ coordinates $(x_i, y_j + \triangle y)$, i.e. $\mathbf{b} = \triangle y\, \mathbf{j} + f_x(x_i, y_j)\triangle y\, \mathbf{k}$. The area of the piece of tangent plane above R_{ij} is then
(15.6.3)

$$\triangle T_{ij} = |\mathbf{a} \times \mathbf{b}| = ... = \sqrt{1 + f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2}\ \triangle A, \quad \text{where} \quad \triangle A = \triangle x \triangle y$$

is the area of R_{ij}. The area of the surface is hence

$$\text{(15.6.4)} \qquad \text{Area}(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{1 + f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2}\ \triangle A.$$

By the definition of double integral, this is

$$\text{(15.6.5)} \qquad \text{Area}(S) = \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\ dA.$$

Note the similarity with the formula for the arc length of a curve in the form $y = f(x)$.

There is an alternative way to derive this that does not use the fact that the area is given by the magnitude of the cross-product. In fact we can interpret (15.6.3) as that it gives the ratio of the area $\triangle T_{ij}$ of the piece of tangent plane above R_{ij} to the area $\triangle A$ of R_{ij} itself. Geometrically it is clear that this ratio of the areas is exactly one over the cosine of the angle between the normals to the tangent plane and the plane $z = 0$, If $\mathbf{n}$ denotes the normal to the tangent plane then the ratio is $1/\cos\gamma = 1/\mathbf{n}\cdot\mathbf{k}$. Writing the surface as $F(x,y,z) = z - f(x,y)$ the normal is $\nabla F = -f_x\mathbf{i} - f_y\mathbf{j} + 1$ so a unit normal is $\mathbf{n} = \nabla F/|\mathbf{F}|$ and $1/\cos\gamma = \mathbf{n}\cdot\mathbf{k} = \sqrt{f_x^2 + f_y^2 + 1}$.

Ex. Find the area of the surface S: $z = f(x,y) = x^2 + y^2$ below the plane $z = 9$.

Sol. Since $f_x = 2x$ and $f_y = 2y$ and $D = \{(x,y);\, x^2 + y^2 \leq 9\}$ we get

$$\text{Area}(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$

If we introduce polar coordinates we get

$$\text{Area}(A) = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r dr\, d\theta = \int_0^{2\pi} \frac{1}{12}(1+4r^2)^{3/2}\Big|_0^3 d\theta = \frac{\pi}{6}\big((1+4\cdot 9)^{3/2} - 1\big)$$