Lecture 26: Tripple Integrals. We now want to define the integral of a function f over a rectangular box $B = \{(x, y, z); a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$. We divde B into smaller boxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ by dividing the interval $[a, b]$ into ℓ subintervals of length $\triangle x$, $[c, d]$ into m subintervals of length $\triangle y$ and $[r, s]$ into n subintervals of length $\triangle z$. Then we form the Riemann sum

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\triangle V \tag{15.7.1}$$

where $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ is a sample point in B_{ijk} and $\triangle V = \triangle x\,\triangle y\,\triangle z$. We then define the integral to be the limit of the Riemann sum:

$$\iiint_B f(x, y, z)\,dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\triangle V \tag{15.7.2}$$

As for the case of two variables we can write it as iterated integrals (Fubini):

$$\iiint_B f(x, y, z)\,dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz \tag{15.7.3}$$

or one can integrate the different variables in any other order.

Ex. Let $B = \{(x, y, z);\ 1 \le x \le 2,\ 0 \le y \le 1,\ 1 \le z \le 2\}$. Find $\iiint_B x^2yz\,dV$.

Sol. $\iiint_B x^2yz\,dV = \int_1^2 \int_1^2 \int_0^1 x^2yz\,dy\,dz\,dx = ...$

As for double integrals we define the integral of f over a more general bounded region E by finding a large box B containing E and integrating the function that is equal to f in E and 0 outside E over the lager box B.

We now restrict our attention to some special regions. First let us consider a region of **type 1**:

$$E = \{(x, y, z);\ (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\} \tag{15.7.4}$$

where D is the projection of E onto the x-y plane. Then

$$\iiint_B f(x, y, z)\,dV = \iint_D \Big(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\Big)dA \tag{15.7.5}$$

In particular if D is a region of type I in the plane then

$$E = \{(x, y, z);\ a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\} \tag{15.7.6}$$

and

$$\iiint_B f(x, y, z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\,dy\,dz \tag{15.7.7}$$

Note also that one can find the volume by using tripple integrals

$$\text{Volume}(E) = \iiint_E 1\,dV \tag{15.7.8}$$

Ex. Let $E=\{(x,y,z);\ x\geq 0,\ y\geq 0,\ z\geq 0,\ x+2y+3z\leq 1\}$. Find the volume of E

Sol. $E=\{(x,y,z);\ 0\leq x\leq 1,\ 0\leq y\leq (1-x)/2,\ 0\leq z\leq (1-x-2y)/3\}$ so

$$V=\iiint_E 1dV=\int_0^1\int_0^{(1-x)/2}\int_0^{(1-x-2y)/3} dz\,dy\,dx=...$$

Region of type 2: If $E=\{(x,y,z);\ (y,z)\in D,\ u_1(y,z)\leq x\leq u_2(y,z)\}$. then

$$\iiint_B f(x,y,z)\,dV=\iint_D\Big(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\Big)dA \tag{15.7.5}$$

Region of type 3: If $E=\{(x,y,z);\ (x,z)\in D,\ u_1(x,z)\leq x\leq u_2(x,z)\}$ then

$$\iiint_B f(x,y,z)\,dV=\iint_D\Big(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\Big)dA \tag{15.7.5}$$

Ex. Evaluate $\iiint_E\sqrt{x^2+z^2}\,dV$ where E is the region bounded by $y=x^2+x^2$ and the plane $y=4$.

Sol. $E=\{(x,y,z);\ (x,z)\in D,\ x^2+z^2\leq y\leq 4\}$, where $D=\{(x,z);\ x^2+z^2\leq 4\}$.

$$\iiint_E\sqrt{x^2+z^2}\,dV=\iint_D\int_{x^2+z^2}^4\sqrt{x^2+z^2}dy\,dA=\iint_D\big(4-(x^2+z^2)\big)\sqrt{x^2+z^2}\,dA$$

Introducing poolar coordinates in the x-z plane: $x=r\cos\theta$, $z=r\sin\theta$ gives that the integral is:

$$\int_0^{2\pi}\int_0^2(4-r^2)r\,rdr\,d\theta=...=\frac{128\pi}{15}$$