Lecture 2: Section 10.1. Note that any curve int the form (10.1.1) can be written in the form (10.1.2) with $h(x,y) = y - F(x)$. Note also that any curve in the form (10.1.1) can be written in the form (10.1.3) where $f(t) = t$ and $g(t) = F(t)$.

Question: When can we write a curve in the form (10.1.3) in the form (10.1.1)?
Answer: When the function $x = f(t)$ has an inverse $t = f^{-1}(x)$.
Question: When does the function $x = f(t)$ have an inverse $t = f^{-1}(x)$?
Answer: When $f'(t) \neq 0$ for all $a \leq t \leq b$.

Ex. Find a Cartesian equation for the curve given in parametric form by $\begin{cases} x = t^2, \\ y = t^3 \end{cases}$.

Note that the most obvious way would have been to find an inverse to the function $x = t^2$. This function is however not invertible, since solving the equation gives $t = \pm\sqrt{|x|}$. Therefore it is not possible to write the curve in the form $y = F(y)$. In this case one can write it in the form $y = G(x)$ but instead we observe that $x^3 = t^6$ and $y^2 = t^6$ so it follows that $x^3 - y^2 = 0$ which is an equation of the form (10.1.2).

Section 10.2: Tangents. Suppose that a curve that can be written both in the form (10.1.1) and (10.1.3). Then $g(t) = y = F(x) = F(f(t))$ so by the chain rule $g'(t) = F'(f(t))f'(t)$. If $f'(t) \neq 0$ it follows that $F'(x) = g'(t)/f'(t)$, or

$$\text{(10.2.1)} \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

We can hence calculate the slope of the tangent line to the curve; dy/dx directly from the parametric form of the curve without writing the curve as a graph.
If $dy/dt = 0$ but $dx/dt \neq 0$ then $dy/dx = 0$ so the tangent line is horizontal.
If $dx/dt = 0$ but $dy/dt \neq 0$ then $dy/dx = \pm\infty$ so the tangent line is vertical.

Ex. Consider the cycloid, i.e. the curve in ex 10.1.6 in the book: $\begin{cases} x = r(\theta - \sin\theta), \\ y = r(1 - \cos\theta) \end{cases}$.

It follows that $\begin{cases} dx/d\theta = r(\theta - \sin\theta) = 0, & \text{if} \quad \theta = 2n\pi \\ dy/d\theta = r(1 - \cos\theta) = 0, & \text{if} \quad \theta = n\pi \end{cases}$, where n is an integer.
If $\theta = (2n+1)\pi$ then $dy/d\theta = 0$ but $dx/d\theta \neq 0$ so it follows that $dy/dx = 0$.
If $\theta = 2n\pi$ then $dy/d\theta = 0$ and $dx/d\theta = 0$ so we can not directly conclude anything about dy/dx. Instead we will take the limit as $\theta \to 2n\pi + 0$ from above.
It follows from (10.2.1) that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\sin\theta}{1 - \cos\theta}$. Hence by l'Hospital's rule:

$$\lim_{\theta \to 2n\pi + 0} \frac{dy}{dx} = \lim_{\theta \to 2n\pi + 0} \frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta \to 2n\pi + 0} \frac{\cos\theta}{\sin\theta} = +\infty.$$

If $(x(t), y(t))$ denotes the coordinates for the position of a particle at time t, then the derivative $(x'(t), y'(t))$ denotes the velocity **vector** of the particle at time t. The velocity vector points in the direction in which the particle is moving and its magnitude is the speed of the particle.

Ex. Sketch the curve $\begin{cases} x = t(t^2 - 3), \\ y = 3(t^2 - 3) \end{cases}$. The points where $\dfrac{dx}{dt} = 0$ or $\dfrac{dy}{dt} = 0$ are of significant interest so we calculate these: $\begin{cases} dx/dt = 3(t^2 - 1) = 0, & \text{when} \quad t = \pm 1 \\ dy/dt = 6t = 0, & \text{when} \quad t = 0 \end{cases}$.
Points where the curve intersects itself are also of interest. These can not be found from a local analysis of the derivatives. In this case $(x, y) = (0, 0)$ when $t = \pm\sqrt{3}$.

At each point we evaluate $(x(t), y(t))$ we also evaluate $(x'(t), y'(t))$ since this tells us in which direction the curve is going.