Lecture 3: Arc length. In section 8 of the book there was a formula for the arc length L of a curve in the form of a graph (10.1.1): $L = \int_a^b \sqrt{1+(dy/dx)^2}\,dx$. If we now also express the same curve in parametric form (10.1.3), use (10.2.1) and a change of variables, $dx = (dx/dt)dt$, in the integral we obtain:

$$(10.3.1) \qquad L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt.$$

We will now show that (10.3.1) holds for any parametrized curve. Let $\alpha = t_0 < t_1 < ... < t_{i-1} < t_i < ... < t_n = \beta$ and $P_i = (x_i, y_i)$ be the point on the curve when the parameter $t = t_i$. Let ℓ_i be the length of the line segment from P_{i-1} to P_i. Then the length of the polygonal curve with vertices at P_i, for $i=0, ..., n$ is $\sum_{i=1}^n \ell_i$. We now let the number of points, $n+1$, tend to infinity in such a way that the maximum distance between neighboring points $t_i - t_{i-1}$ tend to 0, and we define

$$(10.3.2) \qquad L = \lim_{n\to\infty} \sum_{i=1}^n \ell_i$$

By the Pythagorean theorem

$$(10.3.3) \quad \ell_i = \sqrt{\triangle x_i^2 + \triangle y_i^2}, \qquad \text{where } \triangle x_i = x_i - x_{i-1} \text{ and } \triangle y_i = y_i - y_{i-1}.$$

Since by the linear approximation $\triangle x_i = f(t_i) - f(t_{i-1}) \sim f'(t_i)\triangle t_i$ and $\triangle y_i = g(t_i) - g(t_{i-1}) \sim g'(t_i)\triangle t_i$, where $\triangle t_i = t_i - t_{i-1}$, it follows that

$$(10.3.4) \qquad \ell_i \sim \sqrt{f'(t_i)^2 + g'(t_i)^2}\triangle t_i$$

However, (10.3.2) with ℓ_i given by (10.3.4) is nothing but the definition of the integral (10.3.1) through using Riemann sums.

One can show that (10.3.1) is independent of the choice of parametrization. Another way to motivate (10.3.1) is as follows. If $s(t)$ denotes the arc length along the curve from α to t, then essentially by the argument leading to (10.3.4)

$$(10.3.5) \qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

With this notation we can symbolically write the integral (10.3.1)

$$(10.3.6) \qquad L = \int ds$$

Ex. Find the arc length of the unit circle.

Sol. 1 $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Then
$L = \int_0^{2\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{2\pi} \sqrt{(\sin t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$

Sol. 2 $x = \cos 2t$, $y = \sin 2t$, $0 \le t \le \pi$. Then
$L = \int_0^{\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{\pi} \sqrt{(2\sin 2t)^2 + (2\cos 2t)^2}\,dt = \int_0^{\pi} 2\,dt = 2\pi.$

Conclusion The arc length is independent of the choice of parametrization, i.e. it doesn't depend on how fast we travel along the curve.

Area of a surface of revolution In section 8.2 there was a formula for the are of a surface of revolution, of a curve given as a graph. Consider the surface in $x - y - z$ space obtained by rotating a curve in the $x - y$ plane around the x-axis. If the curve is in the form of a graph $y = F(x)$, for $a \leq x \leq b$, where $F(x) \geq 0$, then the area is given by

$$A = \int_a^b 2\pi y \sqrt{1 + (dy/dx)^2}\, dx \tag{10.3.7}$$

The proof of this formula uses that the area of the piece of the surface between x and $x + \triangle x$ is approximately $2\pi y \triangle s$, where y is the distance from the curve to the x-axis and $\triangle s$ is the length of the arc of the curve between x and $x + \triangle x$. Since the curve can be approximated by a line of slope dy/dx it follows as before in this section that $\triangle s \sim \sqrt{\triangle x^2 + \triangle y^2} \sim \sqrt{1 + (dy/dx)^2}\triangle x$. (10.3.7) follows from this by approximating the integral by a Riemann sum.

If the curve is also given in parametric form then we can as previously make the change of variable $x = f(t)$, $dx = (dx/dt)dt$ and $dy/dx = (dy/dt)/(dx/dt)$ to obtain

$$A = \int_\alpha^\beta 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt \tag{10.3.8}$$

Using essentially the same argument as for the arc length one can show that (10.3.8) holds for any parametrized curve. Furthermore with ds given by (10.3.5) we can symbolically write

$$A = \int 2\pi y\, ds \tag{10.3.9}$$

Ex. Find the area of the surface of the unit sphere.

Sol. The sphere can be obtained as the surface of revolution of the part of the unit circle in the upper half plane. This can be parametrized by $x = \cos t$ and $y = \sin t$, where $0 \leq t \leq \pi$. Using the above formula it follows that
(10.3.10)

$$A = \int_0^\pi 2\pi \sin t \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^\pi 2\pi \sin t\, dt = -2\pi \cos t\Big|_0^\pi = 4\pi$$