Lecture 4: 10.4 Polar coordinates. Polar coordinates (r, θ) corresponds to rectangular coordinates (x, y) by the relation $\begin{cases} x = r\,\cos\theta, \\ y = r\,\sin\theta \end{cases}$ and $r^2 = x^2 + y^2$, $\tan\theta = y/x$. Note that (r, θ) corresponds to the same point in the $x - y$ plane as $(r, \theta + 2n\pi)$ and $(-r, (2n+1)\pi)$, where n is an integer. A curve in polar coordinates:

$$(10.4.1) \qquad x = r\,\cos\theta, \qquad y = r\,\sin\theta, \qquad r = f(\theta)$$

is just a special case of a parametrized curve.

Ex. Sketch the curve $r = 2\sin\theta$ and find its Cartesian eq.

Sol. The curve is a circle of radius 1 centered at $(0, 1)$. In fact, multiplying by r gives $r^2 = 2r\,\sin\theta$ which is the same as $x^2 + y^2 = 2y$ which if we complete the square is equivalent to $x^2 + (y-1)^2 = 1$.

Ex. Sketch the curve $r = \sin 2\theta$.

Sol. The curve is a four-leaved rose.

Ex. Sketch the curve $r = 2/(1 - \cos\theta)$ and find its Cartesian eq.

Sol. The curve is a parabola. In fact, the equation says that $r - r\cos\theta = 2$ so $r^2 = (2 + r\cos\theta)^2$ or $x^2 + y^2 = (2 + x)^2$ so $y^2 = 4(x + 1)$.

Tangents to polar curves To calculate the slope of the tangent we use the formula (10.2.1):

$$(10.4.2) \qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$$

Ex. Find the tangent to the parabola $r = 2/(1 - \cos\theta)$ at $\theta = \pi/2$.

Sol. When $\theta = \pi/2$, $r = 2$ and $dr/d\theta = -2\sin\theta/(1 - \cos\theta)^2 = -2$ so it follows that $dy/dx = -2/(-2) = 1$.

Ex. Find the points on the curve $r = 2\sin\theta$ where the tangent line is either horizontal or vertical.

Sol. First we note that $dr/d\theta = 2\cos\theta$. Hence $dy/d\theta = 4\sin\theta\cos\theta = 2\sin 2\theta = 0$, if $\theta = n\pi/2$ and $dx/d\theta = 2(\cos^2\theta - \sin^2\theta) = 2\cos 2\theta = 0$, if $\theta = n\pi/2 + \pi/4$, where n is an integer. It follows that $dy/dx = 0$ when $\theta = n\pi/2$ so then the tangent line is horizontal. It also follows that $dy/dx = \pm\infty$ when $\theta n\pi/2 + \pi/4$ so then the tangent line is vertical.

10.5: Arc length in polar coordinates. By plugging in the polar curve (10.4.1) in the formula for the arc length (10.3.1) and calculating $(dx/\theta)^2 + (dy/\theta)^2 = \big((dr/d\theta)\sin\theta + r\cos\theta\big)^2 + \big((dr/d\theta)\cos\theta - r\sin\theta\big)^2 = ... = r^2 + (dr/\theta)^2$ we get

$$(10.4.3) \qquad L = \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$$

Ex. Find the arc length of the polar curve $r = 2\sin\theta$, where $0 \le \theta \le \pi$.

Sol. We have $L = \int_0^\pi \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta = 2\int_0^\pi d\theta = 2\pi$.

10.6: Conic sections. By intersecting a cone with a plane one gets 3 types of curves: ellipse $x^2/a^2 + y^2/b^2 = 1$, parabola $x^2 = 4py$ and hyperbola $y^2/b^2 - x^2/a^2 = 1$.