Lecture 5: 12.1 Coordinate systems. A point P in the plane can be identified with and ordered pair of real numbers (a, b), called coordinates, as follows. We pick an origin O and from it draw two perpendicular directed lines, called coordinate axes. One line is horizontal with positive direction to the right, called the x-axis, and the other line is vertical with positive direction going up, called the y-axis. We reach the point $P(a, b)$ by going a units from the origin in the x direction and b units in the y direction. The coordinate system so obtained is called rectangular or Cartesian coordinates. (There are other coordinate systems e.g. polar coordinates.) We remark that it was essential to decide that the x-axis was directed to the right and the y-axis was directed up. This is called positive orientation. If we change the direction of the y-axis so it points down then the sign of the y-coordinate change. The positive orientation of the $x-y$ plane is determined by that if we turn a quarter of a turn counter clockwise from the positive x-axis we reach the positive y-axis.

A point P in space can be represented by an ordered triple (a, b, c) of real numbers as follows. We pick an origin O and from it draw three perpendicular directed lines, called coordinate axes. We think of two of the directions, the x-axis and the y-axis as lying in the horizontal plane and the z-axis as being directed vertically upward. Furthermore, seen from the side of the positive z-direction, we ask that the x-axis and y-axis form a positive orientation of the horizontal plane. This is called a positively oriented coordinate system. It is also given by the right-hand-rule; If you curl your fingers of the right hand around the z-axis in the direction of a 90 degree counter clockwise turn from the positive x-axis to the positive y-axis the your thumb should point in the positive z-axis. Imagine a room with the origin in the opposite corner of where you are standing. The floor is formed by the x-axis and y-axis, the wall to the left is formed by the x-axis and z-axis and the wall to the right is formed by the $y-$axis and z-axis. The point $P(a, b, c)$ is reached by going a units from the origin in the x direction, b units in the y direction and then c units in the z direction. It is called the rectangular coordinate system, since it determine a rectangular box with edges of length a, b and c.

Ex. Note that one equation $y = 5$ in two dimensions determine a line, whereas in three dimensions it determines a plane.

Distance formula In two dimensions the distance between a point $P_1(x_1, y_1)$ and a point $P_2(x_2, y_2)$ is given by the Pythagorean Theorem:

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{12.1.1}$$

In three dimensions the distance between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{12.1.2}$$

For the proof we note that we are looking for the length d of the diagonal of a rectangular box with edges of length $a = |x_2 - x_1|$, $b = |y_2 - y_1|$ and $c = |z_2 - z_1|$. The length of the diagonal in the two dimensional rectangle with sides a and c is by the Pythagorean theorem $\sqrt{a^2 + b^2}$. The diagonal of the box is then also the diagonal of a two dimensional rectangle with one side of length $\sqrt{a^2 + b^2}$ and the other side of length c so by the Pythagorean theorem $d = \sqrt{(\sqrt{a^2 + b^2})^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$.

Ex. Find the eq. for a sphere with radius r and center at $C(h, k, l)$.

Sol. By definition, the sphere is the set of points $P(x, y, z)$ at distance r from $C(h, k, l)$, $|C P| = r$ or $|C P|^2 = r^2$. By (12.1.2) this is given by

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2. \tag{12.1.3}$$

12.2: Vectors. Imagine a particle moving along a path or yourself driving on a car on a road. At each point the velocity $\mathbf{v}$ has both a **magnitude** (the speed that is measured be the cars meter) and a **direction** in which it is going (that you need a compass to measure). Such a quantity is called a **vector**. It is represented by a directed line segment or arrow. In a sense two vectors are the same if they have the same magnitude and direction even if they start from different points.

Def. A two dimensional **vector a** is an ordered pair of real numbers $\langle a_1, a_2\rangle$ called components. A three dimensional **vector a** is an ordered triple of real numbers $\langle a_1, a_2, a_3\rangle$ called components. A representative of the vector $\mathbf{a} = \langle a_1, a_2\rangle$ is a directed line segment from a point $A(x, y)$ to a point $B(x+a_1, y+a_2)$. A particular representative is the position vector of the point $P(a_1, a_2)$ which is the directed line segment from the origin O to P.

Length The length or magnitude of a 2-dim. vector $\mathbf{a} = \langle a_1, a_2\rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$. The length of a 3-dim. vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{12.2.1}$$

Vector addition The vector addition of two 2-dim. vectors $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ is $\mathbf{a}+\mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle$. The vector addition of two 3-dim. vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is

$$\mathbf{a}+\mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle. \tag{12.2.2}$$

Vector addition can also be represented **geometrically**, as apposed to the above algebraic definition, by either the triangle law or the parallelogram law. The triangle law is as follows. Let the vector **a** start at the origin and let the vector **b** start from where **a** ends. The vector so formed from the origin to the end of **b** is the vector addition $\mathbf{a} + \mathbf{b}$. The parallelogram law is as follows. Let the vectors **a** and **b** start from the origin and form the parallelogram with sides given by **a** and **b**. The diagonal of the parallelogram is $\mathbf{a} + \mathbf{b}$.

Ex. Vector addition of forces If we have two forces $\mathbf{F}$ and $\mathbf{F}'$ acting on a particle in different directions then the resulting force on the particle is the vector sum of the two forces $\mathbf{F} + \mathbf{F}'$. In particular the magnitude of the resulting force is strictly less than the sum of the magnitudes of the two forces: $|\mathbf{F}+\mathbf{F}'| \le |\mathbf{F}|+|\mathbf{F}'|$.

Scalar multiplication Let c be a scalar, i.e. just a real number as apposed to a vector. If $\mathbf{a} = \langle a_1, a_2\rangle$ is a 2-dim. vector then the scalar multiplication of **a** by c is given by $c\mathbf{a} = \langle ca_1, ca_2\rangle$. If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is a 3-dim. vector then the scalar multiplication of **a** by c is given by

$$c\mathbf{a} = \langle ca_1, ca_2, ca_3\rangle. \tag{12.2.3}$$

Scalar multiplication $c\mathbf{a}$ can also be represented geometrically. Simply go in the direction of the vector **a** a distance $c|\mathbf{a}|$.