Lecture 6: 12.2: Vectors, continuation of previous lecture.
Last time we introduced **vectors** $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$.
We defined the **length** of a vector $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$,
vector addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$
and **multiplication by a scalar**: $c\mathbf{a} = \langle ca_1, ca_2, ca_3\rangle$.

We now make a couple of further definitions:
The two vectors $\mathbf{a}$ and $\mathbf{b}$ are called **parallel** if $\mathbf{a} = c\,\mathbf{b}$ for some scalar c.
This means that $\mathbf{a}$ points in the same or opposite direction as $\mathbf{b}$ and that $|\mathbf{a}| = |c|\,|\mathbf{b}|$.
In fact $|c\mathbf{b}| = \sqrt{(cb_1)^2 + (cb_2)^2 + (cb_3)^2} = \sqrt{c^2}\sqrt{b_1^2 + b_2^2 + b_3^2} = |c|\,|\mathbf{b}|$.
In particular $-\mathbf{b} = (-1)\mathbf{b}$ is a vector pointing in the opposite direction as $\mathbf{b}$.
The difference of two vectors is now defined by $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.
This also has a geometric interpretation: If $\mathbf{a}$ and $\mathbf{b}$ both start from the origin, then $\mathbf{a} - \mathbf{b}$ is the vector going from the end of $\mathbf{b}$ to the end of $\mathbf{a}$.

The standard **basis** vectors are: $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$ and $\mathbf{k} = \langle 0, 0, 1\rangle$.
Any vector can be written in terms of these: $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.
In fact using the scalar multiplication and vector addition rules we get:
$a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = a_1\langle 1, 0, 0\rangle + a_2\langle 0, 1, 0\rangle + a_3\langle 0, 0, 1\rangle$
$= \langle a_1, 0, 0\rangle + \langle 0, a_2, 0\rangle + \langle 0, 0, a_3\rangle = \langle a_1, a_2, a_3\rangle = \mathbf{a}$

A **unit** vector is a vector who's length is one, e.g. $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.
A unit vector in the direction of the vector $\mathbf{a}$ is given by $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$.
In fact, if $c = 1/|\mathbf{a}|$ then $\mathbf{u} = c\mathbf{a}$ and $|\mathbf{u}| = c|\mathbf{a}| = 1$.

Ex. Let $\mathbf{a} = \langle 3, 2, -1\rangle$ and $\mathbf{b} = \langle 0, 6, 7\rangle$. Find $\mathbf{a} - 2\mathbf{b}$, $|\mathbf{a}|$ and a unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$.
Sol. $\mathbf{a} - 2\mathbf{b} = \langle 3, 2, -1\rangle - 2\langle 0, 6, 7\rangle = \langle 3, 2, -1\rangle + \langle 0, -12, -14\rangle = \langle 3, -10, -15\rangle$, $|\mathbf{a}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$ and $\mathbf{u} = \mathbf{a}/|\mathbf{a}| = \langle 3/\sqrt{14}, 2/\sqrt{14}, -1/\sqrt{14}\rangle$.

12.3:. The dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{12.3.1}$$

It is also called the scalar product or inner product.

Properties of the dot product. A couple of properties are of particular interest:

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \tag{12.3.2}$$
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \tag{12.3.3}$$
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \tag{12.3.4}$$

(12.3.2) follows from the definition of the length of a vector and the dot product.
(12.3.3) follows from the definition of vector addition and the dot product after writing out what each side means in components. It is an expression of that the dot product is linear in its arguments which is important.

The dot product also has a **geometric interpretation** as follows.
Let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta \tag{12.3.5}$$

We could have taken this as definition, but then the linearity (12.3.3) is not obvious.
We must therefore prove (12.3.5). Let $\mathbf{a}$ and $\mathbf{b}$ start from the origin and consider

the triangle with these as two of its edges. The third edge is then given by the vector $\mathbf{a} - \mathbf{b}$. By the law of cosines:

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\,|\mathbf{b}| \cos\theta \tag{12.3.6}$$

On the other hand by using the properties above (12.3.2)-(12.3.4)

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} \tag{12.3.7}$$

Comparing (12.3.6) and (12.3.7) gives (12.3.5).

As a corollary to (12.3.6) we see that we can use it to calculate **angles** θ:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \tag{12.3.8}$$

Ex. Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.
Sol. $|\mathbf{a}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$, $|\mathbf{b}| = \sqrt{5^2 + 3^2 + 2^2} = \sqrt{38}$ and $\mathbf{a} \cdot \mathbf{b} = 2 \cdot 5 + 2 \cdot (-3) + (-1) \cdot 2 = 2$ so $\cos\theta = \mathbf{a} \cdot \mathbf{b}/(|\mathbf{a}|\,|\mathbf{b}|) = 2/(3\sqrt{38})$ and $\theta = \cos^{-1}\big(2/(3\sqrt{38})\big) = 1.46..$ rad or 84 degrees.

We say that two vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** or **orthogonal** if the angle between them $\theta = \pi/2$. Since $\cos(\pi/2) = 0$ this is therefore equivalent to

$$\mathbf{a} \cdot \mathbf{b} = 0 \tag{12.3.9}$$

Ex. $\mathbf{i}$ and $\mathbf{j}$ are orthogonal since $\mathbf{i} \cdot \mathbf{j} = ... = 0$.

Projections. The **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ is defined to be the vector $\mathbf{proj_a}(\mathbf{b})$ parallel to $\mathbf{a}$ such that $\mathbf{b} - \mathbf{proj_a}(\mathbf{b})$ is orthogonal to $\mathbf{a}$, i.e.

$$\mathbf{proj_a}(\mathbf{b}) = c\,\mathbf{a}, \quad \text{for some } c, \quad \text{and} \quad \big(\mathbf{b} - \mathbf{proj_a}(\mathbf{b})\big) \cdot \mathbf{a} = 0. \tag{12.3.10}$$

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the component of $\mathbf{b}$ along $\mathbf{a}$) is defined to be the scalar $\text{comp}_\mathbf{a}(\mathbf{b})$ such that

$$\mathbf{proj_a}(\mathbf{b}) = \text{comp}_\mathbf{a}(\mathbf{b})\,\mathbf{u}, \quad \text{where} \quad \mathbf{u} = \mathbf{a}/|\mathbf{a}| \tag{12.3.11}$$

is a unit vector in the direction of $\mathbf{a}$, i.e. $\text{comp}_\mathbf{a}(\mathbf{b}) = \pm|\mathbf{proj_a}(\mathbf{b})|$ where the sign is the same as the sign of $\mathbf{a} \cdot \mathbf{b}$. It follows that

$$\text{comp}_\mathbf{a}(\mathbf{b}) = |\mathbf{b}| \cos\theta = \mathbf{b} \cdot \mathbf{u} = \mathbf{b} \cdot \mathbf{a}/|\mathbf{a}| \tag{12.3.12}$$

and

$$\mathbf{proj_a}(\mathbf{b}) = \text{comp}_\mathbf{a}(\mathbf{b})\mathbf{a}/|\mathbf{a}| = \big(\mathbf{b} \cdot \mathbf{a}/|\mathbf{a}|^2\big)\mathbf{a} \tag{12.3.13}$$

Ex. Find the scalar and vector projection of $\mathbf{b} = \langle 5, -3, 2 \rangle$ onto $\mathbf{a} = \langle 2, 2, -1 \rangle$.
Sol. $\text{comp}_\mathbf{a}(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}/|\mathbf{a}| = 2/3$ and $\mathbf{proj_a}(\mathbf{b}) = \big(\mathbf{a} \cdot \mathbf{b}/|\mathbf{a}|^2\big)\mathbf{a} = (2/3^2)\langle 2, 2, -1 \rangle$.

The **Physical interpretation** of the dot product is work. If a force $\mathbf{F}$ acts on a particle that is displaced a directed distance $\mathbf{D}$ then the work done by the force on the particle is given by $W = \mathbf{F} \cdot \mathbf{D}$.