Lecture 7: 12.4 The cross product. The **cross product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is defined to be the vector

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2\,,\ a_3b_1 - a_1b_3\,,\ a_1b_2 - a_2b_1\rangle. \tag{12.4.1}$$

The reason for this strange definition is that it is has many useful properties. I order to remember this definition we introduce so called **determinants**.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{12.4.2}$$

Geometrically, this is plus or minus the **area** of the parallelogram in the plane with the vectors $\langle a, b\rangle$ and $\langle c, d\rangle$ as two edges. (This will be shown shortly.)
A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{12.4.3}$$

Geometrically, this is plus or minus the **volume** of the parallelepiped in the plane with the vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ as three edges.

Writing $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ the cross product (12.4.1) is

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \tag{12.4.4}$$

Because of the similarity with (12.4.3), to remember this we symbolically write

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{12.4.5}$$

Note that it follows that $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, i.e. the cross product is anti-commutative. We also note that using (12.4.5) it follows that the so called **scalar triple product**

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{12.4.6}$$

Ex. If $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 2, 4\rangle$ then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{k} \tag{12.4.7}$$
$$= \big(2\cdot 4 - 3\cdot 2\big)\mathbf{i} - \big(1\cdot 4 - 3\cdot(-1)\big)\mathbf{j} + \big(1\cdot 2 - 2\cdot(-1)\big)\mathbf{k} = 2\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

We will now show the geometric significance of the cross product:
Theorem 1 The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = 0, \qquad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0. \tag{12.4.8}$$

It follows that $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ and hence the direction is determined up to a sign, and the side of the plane the vector is on is determined by that the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ form a right handed triple: If your fingers on your right hand curl in the direction of rotation from $\mathbf{a}$ to $\mathbf{b}$, at an angle less than π, then your thumb point in the direction of $\mathbf{a} \times \mathbf{b}$.

Theorem 2 If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \le \theta \le \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta. \tag{12.4.9}$$

It follow from this that the length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$, which also shows that (12.4.2) is the area.

Another consequence is that two vectors are parallel if and only if $\mathbf{a} \times \mathbf{b} = 0$.

Proof of Theorem 1: By (12.4.1) and the definition of the dot product

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (a_2 b_3 - a_3 b_2) a_1 + (a_3 b_1 - a_1 b_3) a_2 + (a_1 b_2 - a_2 b_1) a_3 = \dots = 0 \tag{12.4.10}$$

Proof of Theorem 2: First we note an identity

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \tag{12.4.11}$$

that follows from expanding out both sides. Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ it follows that

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 (1 - \cos^2\theta) = |\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\theta. \tag{12.4.12}$$

Ex. Find the area of the triangle with vertices $P(1,0,-1)$, $Q(0,4,5)$ and $R(3,1,7)$.

Sol. Let $\mathbf{a} = \overline{PQ} = \langle -1,4,6 \rangle$, and $\mathbf{b} = \overline{PR} = \langle 2,1,8 \rangle$. Then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & 6 \\ 2 & 1 & 8 \end{vmatrix} = (32-6)\mathbf{i} - (-8-12)\mathbf{j} + (-1-8)\mathbf{k} = 26\mathbf{i} + 20\mathbf{j} - 9\mathbf{k} \tag{12.4.13}$$

and $A = |\mathbf{a} \times \mathbf{b}|/2 = \sqrt{26^2 + 20^2 + 9^2}/2 = 17.007...$

Volume of a parallelepiped Consider the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Recall that $A = |\mathbf{b} \times \mathbf{c}|$ is the area of the parallelogram with $\mathbf{b}$ and $\mathbf{c}$ as edges. If θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$ then $h = |\mathbf{a}|\,|\cos\theta|$ is the hight of the parallelepiped. Therefore the volume is

$$V = Ah = |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}|\,|\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \tag{12.4.14}$$

Ex. Find the volume of a parallelepiped determined by $\mathbf{a} = \langle 1,-2,0 \rangle$, $\mathbf{b} = \langle -1,4,6 \rangle$ and $\mathbf{c} = \langle 2,1,8 \rangle$.

Sol. By previous example $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \big|(\mathbf{i} - 2\mathbf{j}) \cdot (26\mathbf{i} + 20\mathbf{j} - 9\mathbf{k})\big| = |26 \cdot 1 + (-2) \cdot 20| = 14$.