Lecture 8: 12.5 Equations of Lines and Planes.

Equations of Lines. In the plane a line is determined by knowing a point $P_0(x_0, y_0)$ on the line and the slope m of the line: $x - x_0 = m(y - y_0)$.

In space, a **line** L is determined by a point $P_0(x_0, y_0, z_0)$ on the line and the direction of the line, described a vector $\mathbf{v}$, parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}$ and $\mathbf{r}_0$ be the position vectors of P and P_0. If $\mathbf{a} = \overrightarrow{P_0P}$ is the vector from P_0 to P then $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. Since $\mathbf{a}$ and $\mathbf{v}$ are parallel it follows that $\mathbf{a} = t\mathbf{v}$, for some **parameter** t. Hence

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \tag{12.5.1}$$

which is called the vector equation of L. If we write this in components: $\mathbf{v} = \langle a, b, c\rangle$, $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ we get

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \tag{12.5.2}$$

which is equivalent to the **parametric equations** of a line:

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct \tag{12.5.3}$$

A line is also determined by knowing two points:

Ex. Find the parametric eq. of the line through the points $A(1, 2, 3)$ and $B(4, 5, 4)$.

Sol. Let $\mathbf{v} = \overrightarrow{AB} = \langle 4-1, 5-2, 4-3\rangle = \langle 3, 3, 1\rangle$ and $P_0(1, 2, 3)$. Then the parametric equations of the line is

$$x = 1 + 3t, \qquad y = 2 + 3t, \qquad z = 3 + t \tag{12.5.4}$$

Equations of Planes. A plane is determined by knowing either of the following:
1) A point on the plane and two vectors parallel to the plane in different directions.
2) Three different points in the plane.
3) A point in the plane and a vector perpendicular to the plane.

Let $P_0(x_0, y_0, z_0)$ be a point in a plane and let $\mathbf{n}$ be a vector perpendicular to the plane, called a **normal** to the plane. Let $P(x, y, z)$ be an arbitrary point on the plane and let $\mathbf{r}$ and $\mathbf{r}_0$ be the position vectors of P and P_0. The vector $\mathbf{r} - \mathbf{r}_0 = \overrightarrow{P_0P}$ is then parallel to the plane. Since the normal is perpendicular to every vector in the plane it follows that

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{12.5.5}$$

This is called the vector equation of the plane. If we write this in components: $\mathbf{v} = \langle a, b, c\rangle$, $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ we get

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0 \tag{12.5.6}$$

or if we multiply together we get the **equation of a plane**:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{12.5.7}$$

A plane is also determined by knowing three points in the plane:
Ex. Find the eq. of the plane through the points $A(0,-2,-4)$, $B(1,0,3)$, $C(4,1,6)$.
Sol. Let $\mathbf{a} = \overrightarrow{AB} = \langle 1,2,7\rangle$, $\mathbf{b} = \overrightarrow{AC} = \langle 4,3,10\rangle$. A normal vector is given by

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 7 \\ 4 & 3 & 10 \end{vmatrix} = -1\mathbf{i} + 18\mathbf{j} - 5\mathbf{k} \tag{12.5.8}$$

A point in the plane is $P_0(0,-2,-4)$ so the eq. of the plane is:

$$(-1)(x-0) + 18(y-(2)) - 5(z-(-4)) = 0. \tag{12.5.9}$$

A line can also be described as the intersection of two planes. In particular, as a special case we can, instead of (12.5.3) write the **symmetric equations** of a line

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \tag{12.5.10}$$

This is obtained by solving each of the parametric equations (12.5.3) for the parameter t and noticing that all the three quantities in (12.5.10) are equal to t.
Notice now that (12.5.10) is in fact the intersection of e.g. the two planes

$$\frac{x-x_0}{a} = \frac{y-y_0}{b}, \qquad \text{and} \qquad \frac{y-y_0}{b} = \frac{z-z_0}{c} \tag{12.5.11}$$

Similarly, the intersection of any two non-parallel planes is a line.
Ex. Find the parametric equations of the line of intersection of the two planes $x+y-z=0$ and $x-y-z=0$.
Sol. Since the line is contained in both the planes it must be perpendicular to the normal of both the planes and hence must be going in the same direction as the vector product of the normals to the planes:

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} \tag{12.5.12}$$

We also need to find a point on the line, say the point where $z=0$. Then $x+y=0$ and $x-y=0$ which has the solution $x=y=0$ so $P_0(0,0,0)$ is on the line of intersection and so the par. eq. for the line is

$$-2x + 2y - 2z = 0 \tag{12.5.13}$$

We also note that the equation of a plane can be written slightly differently

$$ax + by + cz + d = 0 \tag{12.5.14}$$

obtained from (12.5.7) by letting $d = -(ax_0 + by_0 + cz_0)$.

Distance formula. We want to find the distance D from a point $P_1(x_1,y_1,z_1)$ to a plane with equation $ax+by+cz+d=0$. It is clear that the closest point $P(x,y,z)$ in the plane must be on the line through P_1 in the direction of the normal $\mathbf{n} = \langle a,b,c\rangle$. Let $P_0(x_0,y_0,z_0)$ be a point on the plane and $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$. Then $\mathbf{proj_n}(\mathbf{b})$, the projection of $\mathbf{b}$ onto the normal $\mathbf{n}$, is in fact equal to $\overrightarrow{PP_1}$ so (12.4.15)

$$D = \frac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} = \frac{|a(x_1-x_0) + b(y_1-y_0) + c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$