Lecture 9: 12.5 cont.. Recall the equation of a plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{12.5.14}$$

which also can be written

$$ax + by + cz + d = 0 \tag{12.5.15}$$

by letting $d = -(ax_0 + by_0 + cz_0)$.

Distance formula. We want to find the distance D from a point $P_1(x_1, y_1, z_1)$ to a plane with equation $ax+by+cz+d = 0$. It is clear that the closest point $P(x, y, z)$ in the plane must be on the line through P_1 in the direction of the normal $\mathbf{n} = \langle a, b, c\rangle$. Let $P_0(x_0, y_0, z_0)$ be a point on the plane and $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$. Then $\mathbf{proj_n}(\mathbf{b})$, the projection of $\mathbf{b}$ onto the normal $\mathbf{n}$, is in fact equal to $\overrightarrow{PP_1}$ so

(12.5.16)

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

12.6: Cylinders and quadratic surfaces. In this lecture we will learn to identify and sketch some simple types of surfaces called cylinders and quadratic surfaces.

Cylinders A cylinder is a surface that consists of all lines that are parallel to a given line and pass through a given plane curve.

If a curve, say in the x-y plane is given by $h(x, y) = 0$, then the surface consisting of all lines parallel to the z-axis that intersects the x-y plane in the curve $h(x, y) = 0$ is a cylinder. In other words, it is a set of points

$$\{(x, y, z);\ h(x, y) = 0\}. \tag{12.6.1}$$

If a curve, say in the y-z plane is given by $z = f(y)$, then the surface consisting of all lines parallel to the x-axis that go through the curve is a cylinder. It is a set

$$\{(x, y, z);\ z = f(y)\}. \tag{12.6.2}$$

However, also other equations not just involving two variables can be cylinders.

You **Sketch** the cylinders by drawing the curves which are the intersections of the surface with planes that are parallel to one of the coordinate planes and drawing lines between these curves.

Ex. Identify and Sketch the surfaces (a) $x^2+y^2=1$ (b) $z = -y^3/8$ (c) $z = -y/4-x^2$.

Sol. To see that (c) is a cylinder we note that the lines defined by the two equations $x = a$ and $z + y/4 = a^2$ lie on the surface, for any constant a.

Quadratic surfaces. A Quadratic surface is a set of (x, y, z) satisfying a quadratic equation, .i.e. it is a zero set of a polynomial of degree two in (x, y, z):

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0 \tag{12.6.3}$$

By translations and rotations one can reduce it to either of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0, \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0 \tag{12.6.4}$$

Examples:
Ellipsoid: $z^2 + \frac{4}{9}x^2 + \frac{1}{4}y^2 = 4.$
Elliptic Paraboloid: $z = \frac{1}{2}x^2 + \frac{1}{2}y^2.$
Hyperbolic Paraboloid: $z = \frac{1}{2}y^2 - \frac{1}{2}x^2.$
Circular Cone: $\frac{1}{3}z^2 = x^2 + y^2.$
Hyperboloid of one sheet: $\frac{1}{3}z^2 = x^2 + y^2 - 1.$
Hyperboloid of two sheet: $\frac{1}{3}z^2 = x^2 + y^2 + 1.$

Several of the surfaces above are of the form (12.6.4) with $A = B$, i.e.

$$A(x^2 + y^2) + Cz^2 + J = 0, \qquad \text{or} \qquad A(x^2 + y^2) + Iz = 0 \tag{12.6.5}$$

In fact, only the hyperbolic paraboloid is not at all of this form. (Also the ellipsoid would have been on this form if we had stretched the x-axis.)

The form (12.6.5) is the form of a **surface of revolution** around the z-axis, i.e.

$$z = F(x^2 + y^2) \tag{12.6.6}$$

This surface depends on x and y through x^2+y^2, which is the square of the distance to the z-axis. A way to **sketch** such a surface is to first draw the curve which is the intersection of the surface with the y-z coordinate plane and then think of this curve as rotated around the z-axis, by drawing a few circles which are the intersection of the surface with the coordinate planes parallel to the x-y plane, where $z =$ const..

As a general rule, when you sketch a surface it is good to **draw the intersections with planes parallel to the coordinate planes**. In particular to draw the hyperbolic paraboloid (or saddle) $z = y^2/2 - x^2/2$ one draws the intersection with the y-z plane, when $x = 0$ then $z = y^2$, and the intersection with the x-z plane, when $y = 0$ then $z = -x^2$.

As remarked above, any surface of the form (12.6.3) can be translated and rotated to a surface of the form (12.6.4).

Ex. Classify and draw the quadratic surface $x^2 + 2z^2 - 6x - y + 10 = 0.$
Sol. By completing the square we get

$$y - 1 = (x - 3)^2 + 2z^2 \tag{12.6.7}$$

which is an elliptic paraboloid translated and rotated, i.e. the z and y axis have interchanged roles.