1. Consider the curve defined parametrically by the equations

$$x(t) = 1 - t^2 \qquad y(t) = 2t^3 + 1$$

for $0 \le t \le 1$.

(a) (10 pts.) Calculate the area between the curve and the x-axis.

$$\begin{aligned} \text{Area} &= \int_\beta^\alpha y(t)x'(t)\, dt \\ &= \int_1^0 (2t^3+1)(-2t)\, dt = \int_1^0 (-4t^4 - 2t)\, dt = -\frac{4}{5}t^5 - t^2\Big|_1^0 = 9/5 \end{aligned}$$

Notice that the lower limit of the integral corresponds to the leftmost point on the curve and the upper limit corresponds to the rightmost point on the curve.

(b) (10 pts.) Calculate the length of the curve.

$$\begin{aligned} \text{Length} &= \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \\ &= \int_0^1 \sqrt{[-2t]^2 + [6t^2]^2}\, dt = \int_0^1 \sqrt{4t^2 + 36t^4}\, dt = \int_0^1 2t\sqrt{1+9t^2}\, dt \end{aligned}$$

Using the substitution $u = 1 + 9t^2$, we see that

$$\int_0^1 2t\sqrt{1+9t^2}\, dt = \frac{2}{27}(1+9t^2)^{3/2}\Big|_0^1 = \frac{2}{27}(10^{3/2} - 1)$$

2. Consider the curve defined parametrically by the equations

$$x(\theta) = 2\theta - \pi\sin\theta \qquad y(\theta) = 2 - \pi\cos\theta$$

for $-\pi \le \theta \le \pi$.

(a) (10 pts.) Find the point where the curve intersects itself.

Using the table to the right, we see that the curve intersects itself at the point $(0, 2)$, corresponding to $\theta = \pm\pi/2$.

θ	$x(\theta)$	$y(\theta)$
$-\pi$	-2π	$2+\pi$
$-\pi/2$	0	2
0	0	$2-\pi$
$\pi/2$	0	2
π	2π	$2+\pi$

(b) (10 pts.) Write down the equation for each line tangent to the curve at the point you found in part (a).

The slope of the tangent line is given by $y'(\theta)/x'(\theta)$ where $x'(\theta) = 2 - \pi\cos\theta$ and $y'(\theta) = \pi\sin\theta$. For $\theta = \pi/2$, the slope of the tangent line is $\pi/2$ and the equation of the tangent line is $y = \frac{\pi}{2}x + 2$. For $\theta = -\pi/2$, the slope of the tangent line is $-\pi/2$ and the equation of the tangent line is $y = -\frac{\pi}{2}x + 2$.

3. (a) (10 pts.) Find the equation of the plane which goes through the points $(1, 1, -1)$, $(0, 1, 0)$ and $(1, -1, 0)$.
   Let $u = \langle -1, 0, 1\rangle$ be the vector that goes from $(1, 1, -1)$ to $(0, 1, 0)$. Let $v = \langle 1, -2, 0\rangle$ be the vector that goes from $(0, 1, 0)$ to $(1, -1, 0)$. Since $u \times v = \langle 2, 1, 2\rangle$ is orthogonal to the plane, the equation of the plane is of the form

$$2x + y + 2z = d.$$

   To solve for d, plug in any one of the three points given. Using the point $(0, 1, 0)$, we see that $d = 1$.

   (b) (10 pts.) Compute the perpendicular distance from the point $(1, 1, 1)$ to the plane described in part (a).
   Let w be a vector that goes from any point on the plane to the point $(1, 1, 1)$. Using the point $(1, 1, -1)$, $w = \langle 0, 0, 2\rangle$. Therefore the perpendicular distance is given by

$$\frac{|(u \times v) \cdot w|}{|u \times v|} = \frac{4}{\sqrt{4+1+4}} = 4/3$$

   where u and v are the same as in the solution to part (a).

4. Let $A = (2, 1, -1)$, $B = (3, 0, -2)$, $C = (3, 2, 1)$, and $D = (-2, 0, 1)$.

   (a) (10 pts.) Find the area of the parallelogram that has AB and AC as adjacent sides.
   Let $u = \langle 1, -1, -1\rangle$ be the vector from A to B and let $v = \langle 1, 1, 2\rangle$ be the vector from A to C. Therefore

$$\text{Area} = |u \times v| = |\langle -1, -3, 2\rangle| = \sqrt{1+9+4} = \sqrt{14}.$$

   (b) (10 pts.) Find the volume of the parallelepiped that has edges AB, AC, and AD.
   Let $w = \langle -4, -1, 2\rangle$ be the vector from A to D. Therefore

$$\text{Volume} = |(u \times v) \cdot w| = |\langle -1, -3, 2\rangle \cdot \langle -4, -1, 2\rangle| = |4 + 3 + 4| = 11.$$

   where u and v are the same as in the solution to part (a).

5. (a) (10 pts.) Find a unit vector which is orthogonal to $\langle 1, -1, -1\rangle$ and $\langle 1, 1, 2\rangle$.
   Notice that these are exactly the vectors used to answer part (a) of the previous question. Thus, we will start with the vector $\langle -1, -3, 2\rangle$. To create a unit vector, we simply divide $\langle -1, -3, 2\rangle$ by it's length, which was also computed in the previous question. Therefore $\frac{1}{\sqrt{14}}\langle -1, -3, 2\rangle$ is a unit vector orthogonal to both $\langle 1, -1, -1\rangle$ and $\langle 1, 1, 2\rangle$.

   (b) (10 pts.) Find the line of intersection of the two planes

$$x - y - z = 2 \qquad x + y + 2z = -2$$

   Notice that the point $(0, -2, 0)$ lies on both planes. Also note that the normal vectors of the two planes are precisely the vectors from part (a). Therefore the direction of the line is parallel to the vector $\langle -1, -3, 2\rangle$ and the line of intersection is given by :

$$x(t) = -t \qquad y(t) = -2 - 3t \qquad z(t) = 2t$$