Solutions for Math 21C Midterm II, Fall 02, Lindblad.

1. (a) Velocity $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3t^{1/2}/2\mathbf{k}$ and speed $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + (3/2)^2 t} = \sqrt{1 + (3/2)^2 t}$.
Hence $\mathbf{r}'(\pi) = -\mathbf{j} + (3/2)\sqrt{\pi}\mathbf{k}$ and $|\mathbf{r}'(\pi)| = \sqrt{1 + (3/2)^2\pi}$.
(b) The distance traveled is the arc length:

$$\int_0^{10} |\mathbf{r}'(t)|\,dt = \int_0^{10} \sqrt{1 + (3/2)^2 t}\,dt = (1+(\tfrac{3}{2})^2 t)^{3/2}(\tfrac{2}{3})^3\Big|_0^{10} = \big((1+(\tfrac{3}{2})^2 10)^{3/2} - 1\big)(\tfrac{2}{3})^3.$$

2. (a) $\nabla F(x,y) = \langle 3x^2 - 2x, 2y - 1\rangle$.
(b) Unit vector in the direction: $\mathbf{u} = \langle 1,2\rangle/|\langle 1,2\rangle| = \langle 1,2\rangle/\sqrt{5}$. $\nabla F(2,3) = \langle 8,5\rangle$.
The directional derivative $D_{\mathbf{u}}F(2,3) = \nabla F(2,3)\cdot\mathbf{u} = \langle 8,5\rangle\cdot\langle 1,2\rangle/\sqrt{5} = 18/\sqrt{5}$.
(c) By the chain rule $h'(t) = \nabla F(\mathbf{r}(t))\cdot\mathbf{r}'(t)$ so $h'(0) = \nabla F(1,0)\cdot\langle 1,1\rangle = \langle 1,-1\rangle\cdot\langle 1,1\rangle = 0$

3. (a) $\langle f_x, f_y\rangle = \langle x, 3y\rangle/\sqrt{x^2+3y^2}$ so $\langle f_x(1,1), f_y(1,1)\rangle = \langle 1,3\rangle/2$ and the equation of the tangent plane is $z - 2 = (x-1)/2 + 3(y-1)/2$.
(b) $z - 2 = 0.1/2 + 3\cdot 0.2/2 = 0.35$ and the linear approximation is $z = 2 + 0.35 = 2.35$.

4. $f_x = 3x^2 - 2x = 0$ and $f_y = 2y - 1 = 0$ gives $(x,y) = (0,1/2)$ or $(x,y) = (2/3,1/2)$.
$f_{xx} = 6x - 2$, $f_{yy} = 2$ and $f_{xy} = 0$ and $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4(3x-1)$.
If $(x,y) = (0,1/2)$ then $D = -4 < 0$ so it is a saddle point.
If $(x,y) = (2/3,1/2)$ then $D = 4 > 0$ and $f_{xx} = 2 > 0$ so it is a local min.

5. Let $F(x,y,z) = d(x,y,z)^2 = x^2 + y^2 + (z+2)^2$ be the square of the distance from $(0,0,-2)$ to (x,y,z) and let $G(x,y,z) = z^2 - x^2 - y^2 - 1$.
We want to maximize $F(x,y,z)$ subject to the constraint $G(x,y,z) = 0$ and $z > 0$.
$\nabla F(x,y,z) = \langle 2x, 2y, 2(z+2)\rangle$ and $\nabla G(x,y,z) = \langle -2x, -2y, 2z\rangle$ so we must find all solutions to

$$2x = -\lambda 2x, \qquad 2y = -\lambda 2y, \qquad 2(z+2) = \lambda 2z, \qquad \text{and} \qquad z^2 - x^2 - y^2 - 1 = 0$$

We see that if $\lambda \neq -1$ then the first two equations gives $x = y = 0$ and the third equation can not be solved if $\lambda = 1$: (i) If $\lambda = 1$ no solutions. (ii) $\lambda = -1$ then the first and second equation hold and the third equation gives $z = -1$ so we get $(x,y,-1)$ and $G(x,y,-1) = -x^2 - y^2 = 0$ gives $x = y = 0$ so the point is $(0,0,-1)$ and $F(0,0,-1) = 1$. If $\lambda \neq -1$ and $\lambda \neq 1$ then $(x,y,z) = (0,0,1/(1+\lambda))$ and $G(0,0,1/(1+\lambda)) = 1/(1+\lambda)^2 - 1 = 0$ gives $\lambda = 0$ so the point is $(0,0,1)$ and $F(0,0,1) = 9$. In conclusion the closest point is $(0,0,1)$.