Solutions for simplified version of Math 20C Midterm 2/20, 96, Lindblad.

1. $V = xyz$. $dV = V_x\,dx + V_y\,dy + V_z\,dz = yz\,dx + xz\,dy + xy\,dz$. Since $x = 20$, $y = 10$ and $z = 5$ we get $dV = 50\,dx + 100\,dy + 200\,dz$ and since for the maximum error $dx = dy = dz = 0.2$ we get $dV = 50 \cdot 0.2 + 100 \cdot 0.2 + 200 \cdot 0.2 = 70$.

2. (a) We know that $D_{\mathbf{u}} f(2,3) = \nabla f(2,3) \cdot \mathbf{u} = 5$ and $D_{\mathbf{v}} f(2,3) = \nabla f(2,3) \cdot \mathbf{v} = -2$ If we write $\nabla f(2,3) = \langle a, b\rangle$ the above to equations simply become $\langle a, b\rangle \cdot \langle 1, 0\rangle = a = 5$ and $\langle a, b\rangle \cdot \langle 0, 1\rangle = b = -2$, which gives that $\nabla f(2,3) = \langle 5, -2\rangle$.
(b) The directional derivative $D_{\mathbf{w}} f(2,3) = \nabla f(2,3) \cdot \mathbf{w} = \langle 5, -2\rangle \cdot \mathbf{w} = 0$ in the direction $\langle 2, 5\rangle$.

3. The equation of the tangent plane at the point $(1,2,0)$ is $F_x(1,2,0)(x-1) + F_y(1,2,0)(y-2) + F_z(1,2,0)(z-0) = 0$. Since $\nabla F = \langle F_x, F_y, F_z\rangle = \langle 2x, -2y, 1\rangle$ we have $\nabla F(1,2,0) = \langle 2, -4, 1\rangle$ so the equation of the tangent plane is $2(x-1) - 4(y-2) + (z-0) = 0$.

4. (a) Now $f_x = 6x + 4y = 0$ and $f_y = 24y + 4x = 0$ implies that the only critical point is $(x,y) = (0,0)$. Since $D = f_{xx} f_{yy} - f_{xy}^2 = 6 \cdot 24 - 4^2 > 0$ and $f_{xx} = 6 > 0$ it follows that $(0,0)$ is a local minimum with $f(0,0) = 0$.
(b) We need to find all (x,y) and λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 8$, i.e.

$$6x + 4y = \lambda 2x, \quad 24y + 4x = \lambda 8y, \qquad x^2 + 4y^2 = 8 \tag{2}$$

If we multiply the first equation by y and the second by $x/4$ we get $6xy + 4y^2 = 2\lambda xy = 6xy + x^2$ so $4y^2 = x^2$, i.e. $x = \pm 2y$. Plugging this into the last equation in (2) gives $4y^2 + 4y^2 = 8$ so $y = \pm 1$. We have found four point that satisfy (2); $(x,y) = (2,1)$, $(-2,-1)$, $(-2,1)$, $(2,-1)$. The maximum of f on the curve is $f(2,1) = f(-2,-1) = 32$ and the minimum is $f(-2,1) = f(2,-1) = 16$.
(c) The absolute max is $f(2,1) = f(-2,-1) = 32$ and the absolute min is $f(0,0) = 0$.

5. We want to minimize the distance squared; $d^2 = G(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint that (x,y,z) is on the surface; $F(x,y,z) = z - y^2 + x^2 + 3 = 0$. Using Lagrange multipliers we must find all (x,y,z) and λ such that $\nabla G(x,y,z) = \lambda \nabla F(x,y,z)$ and $F(x,y,z) = 0$, i.e.

$$2x = 2\lambda x, \quad 2y = -2\lambda y, \quad 2z = \lambda, \tag{3}$$

and $F(x,y,z) = z - y^2 + x^2 + 3 = 0$. The first equation says that $\lambda = 1$ or $x = 0$, the second equation says that $\lambda = -1$ or $y = 0$ and the fird equation says that $z = \lambda/2$. It is now natural to divide into 3 cases depending on the value of λ.
(I): If $\lambda = 1$ then the equations (3) simply says that $y = 0$ and $z = 1/2$ so $(x,y,z) = (x, 0, 1/2)$. Furthermore $F(x, 0, 1/2) = 1/2 + x^2 + 3 > 0$ for all x so these points are excluded since they are not on the surface $F(x,y,z) = 0$.
(II) If $\lambda = -1$ then the equations (3) says that $x = 0$ and $z = -1/2$ and $F(0, y, -1/2) = -1/2 - y^2 + 3 = 0$ if $y^2 = 5/2$ so $y = \pm\sqrt{5/2}$ and $G(0, \pm\sqrt{5/2}, -1/2) = 11/4$.
(III) If $\lambda \neq 1$ and $\lambda \neq -1$ then by (3) $x = 0$ and $y = 0$ and $z = \lambda/2$ so we get $(x,y,z) = (0, 0, \lambda/2)$ and $F(0, 0, \lambda/2) = \lambda/2 + 3 = 0$ if $\lambda = -6$ and $G(0,0,-3) = 9 > 11/4$.
In conclusion: The two closest points are $(0, \pm\sqrt{5/2}, -1/2)$.