Lecture 13: 3.1-3.2: Second order linear differential equations. We are now going to study the initial value problem for second order linear differential equations:

$$y'' + p\,(t)\,y' + q(t)\,y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0 \tag{3.1.1}$$

Such equations are likely to show up in physics since Newton's second law: $F = ma$ talks about the acceleration a of a particle, which is the first order derivative of the velocity v but the second order derivative of the position x:

$$m\,a = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = F$$

In the previous example with the falling body it was sufficient to just look at the equation for the velocity since the force only depended on the velocity: $F = F(v)$. However, in general if the force also depends on the position $F = F(x, v)$ then the acceleration has to be thought of as the second order derivative of the velocity.
An example is that of a weight with mass m hanging in a spring. If y is the displacement from the equilibrium position then the force from the spring acting on the mass is $-ky$, where $k > 0$ is called the spring constant. By Newton's second law, $ma = F$, we get $my'' = -ky$ or

$$m\,y'' + k\,y = 0, \tag{3.1.2}$$

Let us also remark that to completely determine he solution of a second order equation we must give initial data for both the function and its derivative. In fact our physical experience tells us that in order to determine the path of a particle we must give both its initial position and initial velocity.

We will focus our attentions on homogeneous equations with **constant coefficients**:

$$a\,y'' + b\,y' + c\,y = 0, \tag{3.1.3}$$

where a, b and c are constant, partly because there are many examples from physics of this form such as (3.1.2) and partly because this is the case for which we will be able to find an explicit expression for the solution.
Based on the fact the general first order equation with constant coefficients

$$\frac{dy}{dt} = ry$$

has a general solution of the form

$$y = Ce^{rt},$$

we guess that the general equation (3.1.3) might have a solution of the form

$$y = e^{rt}$$

for some r to be determined. If we substitute $y = e^{rt}$ into (3.1.3) and use that

$$\frac{d}{dt}\,e^{rt} = re^{rt}, \qquad \frac{d^2}{dt^2}\,e^{rt} = r^2e^{rt}$$

we get

$$a\,y'' + b\,y' + c\,y = ar^2 e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} = 0,$$

if r satisfies the so called **characteristic equation**

$$ar^2 + br + c = 0 \tag{3.1.4}$$

We have hence shown that if r is a root of (3.1.4) then $y = e^{rt}$ satisfies (3.1.3). Since, in general the second degree polynomial (3.1.4) has two roots r_1, r_2 we in fact have found two solutions to (3.1.3), $e^{r_1 t}$ and $e^{r_2 t}$, if $r_1 \neq r_2$.
However, since for a linear equation also a constant times a solution is a solution and the sum of tow solutions is a solution it follows that

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{3.1.5}$$

also is a solution. In fact, if y_1 and y_2 are two solutions to (3.1.3) then so is $y = c_1 y_1 + c_2 y_2$, since

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = ac_1\frac{d^2y_1}{dt^2} + ac_2\frac{d^2y_2}{dt^2} + bc_1\frac{dy_1}{dt} + bc_2\frac{dy_2}{dt} + cc_1y_1 + cc_2y_2$$

$$= c_1\Big(a\frac{d^2y_1}{dt^2} + b\frac{dy_1}{dt} + cy_1\Big) + c_2\Big(a\frac{d^2y_2}{dt^2} + b\frac{dy_3}{dt} + cy_4\Big) = 0$$

If $r_1 \neq r_2$ and they are both real (3.1.5) turns out to be the **general solution** of (3.1.3), i.e. any solution is of this form.
Ex Find the general solution to

$$y'' + 3y' + 2y = 0 \tag{3.1.6}$$

Sol The charateristic equation is $r^2 + 3r + 2 = (r+2)(r+1)$, so the roots are $r_1 = -1$ and $r_2 = -2$. Hence the general solution is

$$y = c_1 e^{-t} + c_2 e^{-2t}. \tag{3.1.7}$$

Ex Find the solution to the initial value problem

$$y'' + 3y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 4. \tag{3.1.8}$$

Sol We already showed that the general solution is given by (3.1.7) and we now have to show that we can determine the constants c_1 and c_2 in (3.1.7) so the initial conditions in (3.1.8) are satisfied. If y is given by (3.1.7) then

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

and hence we must solve

$$y(0) = c_1 + c_2 = 1, \qquad y'(0) = -c_1 - 2c_2 = 4$$

Adding the two equations together we get $-c_2 = 5$ so $c_2 = -5$ and $c_1 = 6$. Hence the solution to (3.1.6) is

$$y = 6e^{-t} - 5e^{-2t}.$$

In a similar way one can show that in general when $r_1 \neq r_2$ and both are real we can choose the constants c_1 and c_2 so (3.1.5) satisfies any initial condition:

$$y(t_0) = y_0, \qquad y'(t_0) = y_1 \tag{3.1.9}$$

In fact it follows from (3.2.5) that

$$y(0) = c_1 + c_2 = y_0, \qquad y'(0) = r_1 c_1 + r_2 c_2 = y_1$$

and it is easy to see that this system in general has a solution if $r_1 \neq r_2$.

3.2 Fundamental Solutions of linear homogeneous equations. Let us first state the existence theorem:
Th Suppose that p, q and g are continuous. Then the initial value problem:

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(0) = y_0, \quad y'(0) = y_1, \tag{3.2.1}$$

has a unique solution $y = \phi(t)$.

The proof basically reduces to the same proof as for the first order differential equation by rewriting it as a system. If we put

$$z = y', \qquad z' = y'' = -p(t)y' - q(t)y + g(t) = -p(t)z - q(t)y + g(t)$$

we get a first order system for (y, z):

$$\begin{aligned} y' &= z, & y(0) &= y_0 \\ z' &= -p(t)z - q(t)y - g(t), & z(0) &= y_1 \end{aligned}$$