Lecture 14: 3.2 Fundamental Solutions of linear homogeneous equations. Most of what we will do in this chapter concerns linear second order differential equations with constant coefficients. However, the results in this section also holds for variable coefficients. Let us first recall the **existence theorem**:

Th 1 Suppose that p, q and g are continuous. Then the differential equation

$$y'' + p\,(t)\,y' + q(t)\,y = 0 \tag{3.2.1}$$

has a unique solution satisfying

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \tag{3.2.2}$$

As was mentioned in last lecture proof basically reduces to the same proof as for the first order differential equation by rewriting it as a system for the vector (y, y'). We also proved the **superposition principle** for linear equations in last lecture:

Th 2 If y_1 and y_2 are solutions to (3.2.1) then so is

$$y = c_1 y_1 + c_2 y_2 \tag{3.2.3}$$

Basically, this section is about finding a condition on y_1 and y_2 that guarantee that all solutions are of the form $y = c_1 y_1 + c_2 y_2$ for some constants c_1 and c_2.
We therefore define the **Wronskian determinant** of two solution by:

$$W = y_1 y_2' - y_1' y_2 \tag{3.2.4}$$

Th 3 If $W(t_0) \neq 0$ then one can find c_1 and c_2 such that (3.2.3) satisfies (3.2.2).

Proof of Th 3 We must solve the following system for the unknowns c_1 and c_2:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_1 \end{aligned} \tag{3.2.5}$$

Each equation represent a line in the (c_1, c_2) plane and the system can be solved as long as the lines intersect and are not parallel. This happens when the vectors $(y_1(t_0), y_2(t_0))$ and $(y_1'(t_0), y_2'(t_0))$ are not parallel, which is the case if (3.2.4) is different from 0 at t_0.

3.4: Complex roots. In this chapter we want to solve the equation

$$ay'' + by' + cy = 0 \tag{3.4.1}$$

where a, b, c are real constants. We saw that $y = e^{rt}$ is a solution to the equation if r is a root of the characteristic equation:

$$ar^2 + br + c = 0 \tag{3.4.2}$$

Let r_1 and r_2 be the roots of (3.4.2). In section 3.1 we saw that if $r_1 \neq r_2$ and they are both real the general solution of (3.4.1) is in fact of the form $y = c_1e^{r_1t} + c_2e^{r_2t}$ for some constants c_1 and c_2. We will now consider the case of complex roots and since a, b, c are real it is well-known that that complex roots come in complex conjugate pairs so unless r_1 and r_2 both are real they must be of the form

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \qquad i = \sqrt{-1} \tag{3.4.3}$$

Analogous to the real case we hope that we get to solutions of the form

$$z_1 = e^{r_1t}, \qquad z_2 = e^{r_2t} \tag{3.4.4}$$

However, we don't even know what e^z is supposed to mean when z is complex. To get a clue we must ask what it is when z is real. One expression is the Taylor series

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{3.4.5}$$

This makes sense also when z is complex since we defined how to multiply and add complex numbers and since the sum is absolutely convergent. However, if we were to use (3.4.5) as a definition we must prove that $e^{z_1+z_2} = e^{z_1}e^{z_2}$ also for complex numbers z_1 and z_2 and showing that we get the series for the sum when we multiply together the series would involve proving some combinatorial identities. Instead we use (3.4.5) to get an expression we take as definition. By (3.4.5);

$$e^{i\mu} = \sum \frac{(i\mu)^k}{k!} = \sum \frac{(-1)^n\mu^{2n}}{(2n)!} + i\sum \frac{(-1)^{n-1}\mu^{2n-1}}{(2n-1)!}$$

But the two series are the Taylor series for $\cos\mu$ respectively $\sin\mu$ so we get

$$e^{i\mu} = \cos\mu + i\sin\mu.$$

Since we want the product rule $e^{\lambda+i\mu} = e^{\lambda}e^{i\mu}$ to hold we define

$$e^{\lambda+i\mu} = e^{\lambda}(\cos\mu + i\sin\mu)$$

for any complex number $\lambda + i\mu$ and hence also for any t

$$e^{rt} = e^{(\lambda+i\mu)t} = e^{\lambda t}\big(\cos(\mu t) + i\sin(\mu t)\big), \qquad r = \lambda + i\mu$$

With this definition it follows from differentiation that also for complex r

$$\frac{d}{dt}\, e^{rt} = r\, e^{rt}$$

Since this was the rule used to prove that (3.4.4) are solutions of (3.4.1) when (3.4.3) are the roots of (3.4.2) it follows that indeed (3.4.4) are solutions to (3.4.1) also when r_1 and r_2 are complex roots of the characteristic polynomial.

We have now found two solution to (3.4.1)

$$\text{(3.4.5)} \qquad z_1 = e^{\lambda t}\big(\cos(\mu t) + i\sin(\mu t)\big), \qquad z_2 = e^{\lambda t}\big(\cos(\mu t) - i\sin(\mu t)\big)$$

In fact the expression for z_2 follows from the one for z_1 by replacing μ by $-\mu$ and using that $\cos(-\mu t) = \cos(\mu t)$ and $\sin(-\mu t) = -\sin(\mu t)$. There is however one remaining problem which is that (3.4.5) are complex but we expect the solutions to (3.4.1) to correspond to some real physical quantity. However

$$\text{(3.4.6)} \qquad y_1 = \frac{z_1 + z_2}{2} = e^{\lambda t}\cos(\mu t), \qquad y_2 = \frac{z_1 - z_2}{2i} = e^{\lambda t}\sin(\mu t)$$

are real solutions to (3.4.1). We can now if we want forget the derivation of these solutions using complex numbers and instead just check that they are solutions. We claim that the general solution of (3.4.1) is of the form

$$y = c_1 y_1 + c_2 y_2$$

In fact, by section 3.2 we only need to check that the Wronskian is non-vanishing

$$W = y_1 y_2' - y_1' y_2 = e^{\lambda t}\cos(\mu t) e^{\lambda t}\big(\mu\cos(\mu t) + \lambda\sin(\mu t)\big)$$
$$- e^{\lambda t}\big(-\mu\sin(\mu t) + \lambda\cos(\mu t)\big)e^{\lambda t}\sin(\mu t) = e^{2\lambda t}\big(\cos^2(\mu t) + \sin^2(\mu t)\big) = e^{2\lambda t} \neq 0$$

Ex Find all solutions to the equation

$$y'' + 2y' + 5y = 0$$

Sol The characteristic polynomial is

$$r^2 + 2r + 5 = (r + 1 + 2i)(r + 1 - 2i)$$

with roots $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$ so the general solution is

$$y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$