Lecture 16: 3.7: Nonhomogeneous equation, variation of parameters. We will now give a general method for finding particular solutions for second order linear differential equations that in principle works for any nonhomogeneous term. Let us first illustrate the idea behind the method by looking at a first order equation:

$$y' + p(t)y = g(t). \tag{3.7.1}$$

Let y_1 be a solution to the corresponding homogeneous equation and set

$$y = uy_1, \qquad \text{where} \quad y_1' + p(t)y_1 = 0,$$

and u is a function to be determined. Then y is a solution of (3.7.1) if

$$y' + p(t)y = u'y_1 + u(y_1' + p(t)y_1) = u'y_1 = g(t),$$

i.e. if $u' = y_1^{-1}g$ or if we integrate

$$u = \int y_1^{-1} g\, dt.$$

This idea, called **variation of parameters**, works also for second order equations:

$$y'' + p(t)y' + q(t)y = g(t) \tag{3.7.2}$$

Let y_1 and y_2 be independent solutions to the homogeneous equation (3.7.2) with $g \equiv 0$ and set

$$y = u_1 y_1 + u_2 y_2 \tag{3.7.3}$$

where u_1 and u_2 are functions to be determined. There are many possible choices and since it is two functions we need two equations to determine them. We have

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$$

The first equation we choose is

$$u_1' y_1 + u_2' y_2 = 0 \tag{3.7.4}$$

so that

$$y' = u_1 y_1' + u_2 y_2'$$

and so that

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'$$

does not contain second order derivatives of u_1 and u_2. Substituting these expressions in (3.7.2) gives

$$y'' + py' + qy = u_1\big(y_1'' + py_1' + qy_1\big) + u_2\big(y_2'' + py_2' + qy_2\big) + u_1'y_1' + u_2'y_2' = g$$

Since y_1 and y_2 are solutions of the homogeneous equations the parentheses vanish and hence we must have

$$u_1' y_1' + u_2' y_2' = g \tag{3.7.5.}$$

The system of two equations (3.7.4)-(3.7.5) for the two unknown u_1' and u_2' can now be solve and the solution is

$$u_1' = -\frac{y_2\, g}{W(y_1, y_2)}, \qquad u_2' = \frac{y_1\, g}{W(y_1, y_2)}, \qquad W(y_1, y_2) = y_1 y_2' - y_1' y_2 \tag{3.7.6}$$

These equations can then in principle be integrated to get u_1 and u_2 and then we get a particular solution y to (3.7.2) from (3.7.3).

Ex Use variations of parameters to find a particular solution to

$$y'' - y' - 2y = 2e^{-t} \tag{3.7.7}$$

Sol First we need to find polynomial is $r^2 - r - 2 = (r+1)(r-2)$ so the general solution to the homogeneous equation is $c_1 y_1 + c_2 y_2$ where $y_1 = e^{-t}$ and $y_2 = e^{2t}$. We are therefore seeking a solution to the inhomogeneous equation of the form

$$y = u_1 e^{-t} + u_2 e^{2t}$$

Then

$$y' = u_1' e^{-t} + u_2' e^{2t} - u_1 e^{-t} + 2u_2 e^{2t}$$

and if require that

$$u_1' e^{-t} + u_2' e^{2t} = 0 \tag{3.7.8}$$

we get

$$y' = -u_1 e^{-t} + 2u_2 e^{2t}$$

and hence

$$y'' = -u_1' e^{-t} + 2u_2' e^{2t} + u_1 e^{-t} + 4u_2 e^{2t}.$$

Substituting into (3.7.7) gives

$$y'' - y' - 2y = u_1(e^{-t} + e^{-t} - 2e^{-t}) + u_2(4e^{2t} - 2e^{2t} - 2e^{2t}) - u_1' e^{-t} + 2u_2' e^{2t} = 2e^{-t}$$

or

$$-u_1' e^{-t} + 2u_2' e^{2t} = 2e^{-t} \tag{3.7.9}$$

The system (3.7.8)-(3.7.9) can easily be solved. Adding the equations together gives $3u_2' e^{2t} = 2e^{-t}$ so $u_2' = 2e^{-3t}/3$ and substituting this into (3.7.8) gives $u_1' = -2/3$. Integrating these equations gives $u_1 = -2t/3 + c_1$ and $u_2 = -2e^{-3t}/9 + c_2$. Hence

$$y = \big(-2t/3 + c_1\big)e^{-t} + \big(-2e^{-3t}/9 + c_2\big)\big)e^{2t}$$

is a particular solution. We can in particular choose $c_1 = c_2 = 0$ in which case

$$y = -\frac{2}{3} t\, e^{-t} - \frac{2}{9} e^{-t}$$

This is a perfectly correct answer. However, we can still find a simpler solution by noting that the last part in fact is a solution of the homogeneous equation and adding a homogeneous solution to a particular solution just gives another particular solution. Therefore we only need the first part.

Ex Use the method of undetermined coefficients to find a particular solution to

$$y'' - y' - 2y = 2e^{-t}$$

Sol The first attempt would be to try Ae^{-t}. However since by the previous example e^{-t} is a solution of the homogeneous equation this would just produce 0. Therefore we try $y = Ate^{-t}$. Then $y' = Ae^{-t} - Ate^{-t}$ and $y'' = -2Ae^{-t} + Ate^{-t}$. Hence

$$y'' - y' - 2y = -2Ae^{-t} + Ate^{-t} - Ae^{-t} + Ate^{-t} - 2Ate^{-t} = -3Ae^{-t} = 2e^{-t}$$

if $A = -2/3$ so $y = -2te^{-t}/3$ is a particular solution.

We remark that although in this case the method of undetermined coefficients was shorter, it only works for functions that are combinations of exponentials, sins, cosines and polynomials. We will now explore yet another general method:

We will now derive what we call the **operator factorization method** that in principle gives a particular solution to any second order constant coefficient non-homogeneous linear differential equation. We write the equation in operator form:

$$L[y] \equiv ay'' + by' + cy = (aD^2 + bD + c)y = g, \qquad D = \frac{d}{dt} \tag{3.7.10}$$

The characteristic polynomial can be factorized $ar^2 + br + c = a(r - r_1)(r - r_2)$ and so can the operator:

$$L[y] = a(D - r_1)(D - r_2)y = g$$

This means that finding a solution y to (3.7.10) is equivalent to finding a solution to the system

$$(D - r_1)z = z' - r_1 z = g, \tag{3.7.11}$$

$$(D - r_2)y = y' - r_2 y = z \tag{3.7.12}$$

The first equation (3.7.11) for z can be solved by multiplying by the integrating factor $e^{-r_1 t}$ and integrating which gives

$$z(t) = \int e^{-r_1 t} g(t)\, dt.$$

Substituting $z(t)$ so obtained into the second equation (3.7.12) for y and similarly multiplying by the integrating factor $e^{-r_2 t}$ and integrating gives

$$y(t) = \int e^{-r_2 t} z(t)\, dt.$$

Ex Use the operator factorization method above to find a particular solution to

$$y'' - y' - 2y = 2e^{-t}$$

Sol The characteristic polynomial is $r^2 - r - 2 = (r - 2)(r + 1)$. Hence by (3.7.10)-(3.7.11):

$$z' - 2z = 2e^{-t}, \qquad y' + y = z$$

The first equation can be solved by multiplying by the integrating factor e^{-2t}

$$\frac{d}{dt}\big(e^{-2t}z\big) = e^{-2t}(z' - 2z) = e^{-2t}2e^{-t} = 2e^{-3t}$$

and integrating

$$e^{-2t}z = -\frac{2}{3}e^{-3t} + c_1$$

so if we pick $c - 1 = 0$

$$z = -\frac{2}{3}e^{-t}$$

We must finally solve

$$y' + y = -\frac{2}{3}e^{-t}$$

Multiplying by integrating factor e^t gives

$$\frac{d}{dt}\big(e^t y\big) = e^t(y' + y) = -\frac{2}{3}$$

and integrating gives

$$e^t y = -\frac{2}{3}t + c_2$$

And if we pick $c_2 = 0$ we get

$$y = -\frac{2}{3}t\, e^{-t}$$