

MATH 20D DIFFERENTIAL EQUATIONS (AND SERIES)

Many phenomena in physics and engineering are described by differential equations which is the main part of the course. First we will however learn about series, in particular Taylor series. These are used to approximate functions by polynomials, that are easier to deal with. We will e.g. show that

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$$

Recall that this function has no anti-derivative in terms of elementary functions, but we can take the anti-derivative of each term and sum up the terms. We can also use a finite number of terms in the series to approximately evaluate the function.

Lectures 1: Section 11.1. An (infinite) **sequence** is an ordered list of numbers

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

Here a_1 is called the first term, a_2 the second, a_n the n th term, and so on.

The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Ex 1 $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}, \quad a_n = \frac{n}{n+1}, \quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\}.$

One can illustrate a sequence either by plotting some terms a_n on the real line, or by plotting the graph of a_n as a function of n , i.e. (n, a_n) in the plane.

It looks like the sequence in Ex. 1 approaches 1 as n becomes large, in fact:

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We say that the sequence in Ex. 1 has the limit 1 as n tends to infinity, and we write this as

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

Def A sequence $\{a_n\}$ has the **limit** L if we can make the terms a_n as close to L as we like by taking n sufficiently large. In this case we write

$$\lim_{n \rightarrow \infty} a_n = L, \quad \text{or} \quad a_n \rightarrow L, \quad \text{as } n \rightarrow \infty$$

If the limit exist we say that the sequence **converges**, otherwise it **diverges**.

The precise meaning of the definition is that L is the limit, if for every $\varepsilon > 0$ there is a corresponding N such that

$$|a_n - L| < \varepsilon, \quad \text{whenever } n > N.$$

The definition says that no matter how small an interval $(L - \varepsilon, L + \varepsilon)$ around L there is an N such that all terms a_n from $n = N$ onwards must lie in the interval.

Ex 2 Show that $(-1)^n$ diverges.

Sol There can not be a limit L since the distance to 1 or -1 would be at least 1.

Def We say that the sequence $\{a_n\}$ **diverges to infinity** and write $\lim_{n \rightarrow \infty} a_n = \infty$, if for every M there is an N such that $a_n > M$, when $n > N$.

Ex 3 The sequences $a_n = n$ and $b_n = (-1)^n n$ both diverge, but $\lim_{n \rightarrow \infty} a_n = \infty$.

The only difference with $\lim_{x \rightarrow \infty} f(x)$, if $f(x)$ is a function, is that n is an integer:

Th If $a_n = f(n)$ and $\lim_{x \rightarrow \infty} f(x)$ exist then $\lim_{n \rightarrow \infty} a_n$ exist.

The limit laws for functions hence hold for sequences; if $\{a_n\}$ and $\{b_n\}$ converge then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n, \quad \lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n,$$

and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}, \quad \text{if } \lim_{n \rightarrow \infty} b_n \neq 0, \quad \lim_{n \rightarrow \infty} c = c$$

Ex 4
$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} (1 + \frac{1}{n})} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} = \frac{1}{1+0} = 1.$$

Ex 5 Show that $\lim_{n \rightarrow \infty} r^n = 0$, if $0 < r < 1$ and $\lim_{n \rightarrow \infty} r^n = \infty$, if $r > 1$.

Pf The exponential function satisfies $\lim_{x \rightarrow \infty} a^x = 0$, if $0 < a < 1$ and $\lim_{x \rightarrow \infty} a^x = \infty$, if $a > 1$.

Squeeze Theorem If $a_n \leq b_n \leq c_n$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ then $\lim_{n \rightarrow \infty} b_n = L$.

Corollary If $\lim_{n \rightarrow \infty} |a_n| = 0$ then $\lim_{n \rightarrow \infty} a_n = 0$.

Pf This follows from the Squeeze Theorem since $-|a_n| \leq a_n \leq |a_n|$.

Ex 6 For what values of r is the sequence $\{r^n\}$ convergent and what is the limit?

Sol By Ex 5 the limit exist and is 0, if $0 < r < 1$, and it is divergent for $r > 1$.

If $r = 0$ the limit exist and is 0, and if $r = 1$ the limit exist and is 1.

If $-1 < r < 0$ then $0 < |r| < 1$ so $\lim_{n \rightarrow \infty} |r^n| = \lim_{n \rightarrow \infty} |r|^n = 0$ and hence $\lim_{n \rightarrow \infty} r^n = 0$.

If $r = -1$ this diverges by Ex 2, and if $r < -1$ it diverges by a similar argument.

Def $\{a_n\}$ is called **increasing** if $a_1 < a_2 < \dots < a_n < a_{n+1} < \dots$ and **decreasing** if $a_1 > a_2 > \dots > a_n > a_{n+1} > \dots$. It is called **monotonic** if it is increasing or decreasing.

Def $\{a_n\}$ is called **bounded above** if $a_n \leq M$ and **bounded below** if $a_n \geq m$, for all n . It is called **bounded** if it is bounded above and below.

Monotonic Sequence Theorem A bounded, monotonic sequence is convergent.

Corollary An increasing sequence either converges or diverges to infinity.

Pf If the sequence is bounded then by the Monotonic Sequence Theorem it converges, else if it is unbounded then it diverges to infinity.

Ex7 Show that $\lim_{n \rightarrow \infty} r^n = 0$, if $0 < r < 1$.

Pf If $0 < r < 1$, then $0 < r^{n+1} = r r^n < r^n < \dots < r < 1$, so it is bounded and decreasing. Hence the limit L exist, and $L = 0$, since $L = \lim_{n \rightarrow \infty} r^n = \lim_{n \rightarrow \infty} r r^{n-1} = r \lim_{n \rightarrow \infty} r^{n-1} = rL$.