Lecture 20: 6.2 More Laplace transforms. Recall the Laplace transform:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{6.2.1}$$

Last time we showed that:

Ex 1 $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, when $s > a$.

Th 1 $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Th 2 $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

We will now apply these rules to get more Laplace transforms:

Ex 2 $\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}$, when $s > 0$.

Sol Using Euler's formula $\sin(bt) = \dfrac{1}{2i}\Big(e^{bit} - e^{-bit}\Big)$ and Ex 1 with a replaced by ib and $-ib$ we get

$$\begin{aligned}\mathcal{L}\{\sin bt\} &= \frac{1}{2i}\mathcal{L}\{e^{bit}\} - \frac{1}{2i}\mathcal{L}\{e^{-bit}\} = \frac{1}{2i}\Big(\frac{1}{s-ib} - \frac{1}{s+ib}\Big)\\ &= \frac{1}{2i}\frac{s+ib-(s-ib)}{(s+ib)(s-ib)} = \frac{1}{2i}\frac{2ib}{s^2-(ib)^2} = \frac{b}{s^2+b^2}\end{aligned}$$

In the same way, or alternatively using Th 1 one can show that;

Ex 3 $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$, when $s > 0$.

Ex 4 A converse of Th 1 is also hold:

Th 3 $\mathcal{L}\{tf(t)\} = -\dfrac{d}{ds}\mathcal{L}\{f(t)\}$.

Pf
$$\begin{aligned}\frac{d}{ds}\mathcal{L}\{f(t)\} &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{d}{ds}\, e^{-st} f(t)\, dt\\ &= \int_0^\infty (-t)e^{-st} f(t)\, dt = \int_0^\infty e^{-st}(-t)f(t)\, dt = \mathcal{L}\{-tf(t)\}.\end{aligned}$$

Ex 5 $\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$, $\quad \mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$, $\quad$ when $s > a$.

Ex 6 $\mathcal{L}\{te^{at}\} = \dfrac{1}{(s-a)^2}$, when $s > a$.

Sol By Th 3 $\mathcal{L}\{te^{at}\} = -\dfrac{d}{ds}\dfrac{1}{s-a} = \dfrac{1}{(s-a)^2}$.

Ex 7 Find the solution of $y'' + y = \sin(2t)$, with initial data $y(0) = 2$, $y'(0) = 1$.
Sol Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation using the formulas in Th 1-2 we get:

$$\mathcal{L}\{y''(t) + y(t)\} = \mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = s^2Y(s) - sy(0) - y'(0) + Y(s)$$
$$= (s^2 + 1)Y(s) - 2s - 1 = \mathcal{L}\{\sin(2t)\} = \frac{2}{s^2 + 4}$$

Hence

$$Y(s) = \frac{1 + 2s}{s^2 + 1} + \frac{2}{(s^2 + 1)(s^2 + 4)}$$

The inverse Laplace transform of the first part is by previous examples $\sin t + 2\cos t$. For the other part we put up partial fractions:

$$\frac{2}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}$$

where A, B, C, D are to be determined. If we put the term on a common denominator again we get

$$\frac{(As + B)(s^2 + 4) + (Cs + D)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}$$
$$= \frac{(A + C)s^3 + (B + D)s^2 + (4A + C)s + 4B + D}{(s^2 + 1)(s^2 + 4)} = \frac{2}{(s^2 + 1)(s^2 + 4)}$$

Hence $A + C = B + D = 4A + C = 0$ and $4B + D = 2$ which gives $A = C = 0$ and $3B = 2$ so $B = 2/3$ and $D = -2/3$. Hence

$$\frac{2}{(s^2 + 1)(s^2 + 4)} = \frac{2}{3}\frac{1}{s^2 + 1} - \frac{2}{3}\frac{1}{s^2 + 4}$$

so

$$Y(s) = 2\frac{s}{s^2 + 1} + \frac{5}{3}\frac{1}{s^2 + 1} - \frac{1}{3}\frac{2}{s^2 + 4}$$

Hence

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t)$$