**Lecture 23: 7.5 Linear systems of differential equations.**

**Ex** Find the solution to the system

$$\mathbf{x}' = A\mathbf{x}, \qquad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -6 & -2 \\ -2 & -9 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix},$$

**Sol** First we want to find the eigenvalues $r$ and eigenvectors $\mathbf{x} \neq 0$:

(7.5.5) $$A\mathbf{x} = r\mathbf{x} \qquad \Leftrightarrow \qquad (A - rI)\mathbf{x} = \mathbf{0}.$$

The eigenvalues are solution of the characteristic equation:

$$0 = \det(A - rI) = \begin{vmatrix} -6-r & -2 \\ -2 & -9-r \end{vmatrix} = (-6-r)(-9-r) - 2^2 = r^2 + 15r + 50$$

$$= r^2 + 2\frac{15}{2}r + \Big(\frac{15}{2}\Big)^2 - \frac{225}{4} + \frac{200}{4} = \Big(r + \frac{15}{2}\Big)^2 - \Big(\frac{5}{2}\Big)^2 = \Big(r + \frac{15}{2} + \frac{5}{2}\Big)\Big(r + \frac{15}{2} - \frac{5}{2}\Big)$$

Hence $\det(A - rI) = (r+5)(r+10)$ so the eigenvalues are $r_1 = -5$ and $r_2 = -10$.
If $r = r_1 = -10$ then (7.5.5) becomes

$$(A - r_1 I)\mathbf{x} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} 4x_1 - 2x_2 = 0 \\ -2x_1 + x_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x_1 = \alpha \\ x_2 = 2\alpha \end{matrix}; \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If $r = r_2 = -5$ then (7.5.5) becomes

$$(A - r_2 I)\mathbf{x} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} -x_1 - 2x_2 = 0 \\ -2x_1 - 4x_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x_1 = 2\beta \\ x_2 = -\beta \end{matrix}; \quad \mathbf{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

where we picked $\alpha = \beta = 1$. We have found eigenvalues and eigenvectors so that $A\mathbf{x}^{(1)} = r_1\mathbf{x}^{(1)}$ and $A\mathbf{x}^{(2)} = r_2\mathbf{x}^{(2)}$. It follows that for any constants $c_1$ and $c_2$

$$\mathbf{x} = c_1 e^{r_1 t}\mathbf{x}^{(1)} + c_2 e^{r_2 t}\mathbf{x}^{(2)}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$. In fact, then

$$\mathbf{x}' = r_1 c_1 e^{r_1 t}\mathbf{x}^{(1)} + r_2 c_2 e^{r_2 t}\mathbf{x}^{(2)}$$

and

$$A\mathbf{x} = c_1 e^{r_1 t} A\mathbf{x}^{(1)} + c_2 e^{r_1 t} A\mathbf{x}^{(2)} = c_1 e^{r_1 t} r_1 \mathbf{x}^{(1)} + c_2 e^{r_1 t} r_2 \mathbf{x}^{(2)}.$$

Since $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ are not parallel they form a basis and we can find $c_1$ and $c_2$ so that

$$\mathbf{x}(0) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

In fact

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 + 2c_2 = a \\ 2c_1 - c_2 = b \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 = (a+2b)/5 \\ c_2 = (2a-b)/5 \end{matrix}$$

and hence

(7.5.6) $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{a+2b}{5}\, e^{-5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2a-b}{5}\, e^{-10t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Note that the solution (7.5.6) tend to $\mathbf{0}$ as $t \to \infty$ for any initial condition, i.e. $\mathbf{0}$ is a **stable equilibrium**. However, to conclude this it would have been sufficient to calculate the eigenvalues and note that both are negative.

The **Direction field** and **phase portrait** are pictures in the $x_1x_2$-plane. By evaluating and plotting the vector $A\mathbf{x}$ starting at a number of points $\mathbf{x}$ we get the direction field and the phase portrait is obtained by also drawing a few solution curves which are tangential to the direction fields. In particular if we do this in the above example we will see that all solution curves go towards the origin $\mathbf{0}$.