Lecture 24: 7.6 Complex eigenvalues.
Ex Find the solution to the system

$$\mathbf{x}' = A\mathbf{x}, \qquad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix},$$

Sol 1 First we want to find the eigenvalues r and eigenvectors $\mathbf{x} \neq 0$:

(7.6.1) $$A\mathbf{x} = r\mathbf{x} \qquad \Leftrightarrow \qquad (A - rI)\mathbf{x} = \mathbf{0}.$$

The eigenvalues are solution of the characteristic equation:

$$0 = \det(A - rI) = \begin{vmatrix} -1-r & -2 \\ 2 & -1-r \end{vmatrix} = (-1-r)^2 + 2^2 = (-1-r-2i)(-1-r+2i)$$

so the eigenvalues are $r_1 = -1 - 2i$ and $r_2 = -1 + 2i$, where $i = \sqrt{-1}$.
If $r = r_1 = 1 - 2i$ then (7.6.1) becomes

$$(A - r_1 I)\mathbf{x} = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} 2ix_1 - 2x_2 = 0 \\ 2x_1 + 2ix_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x_1 = \alpha \\ x_2 = \alpha i \end{matrix}; \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

If $r = r_2 = 1 + 2i$ then (7.6.1) becomes

$$(A - r_2 I)\mathbf{x} = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} -2ix_1 - 2x_2 = 0 \\ 2x_1 - 2ix_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x_1 = \beta \\ x_2 = -\beta i \end{matrix}; \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Then for any complex constants c_1 and c_2

$$\mathbf{x} = c_1 e^{r_1 t}\mathbf{x}^{(1)} + c_2 e^{r_2 t}\mathbf{x}^{(2)}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$. In fact, then

$$\mathbf{x}' = r_1 c_1 e^{r_1 t}\mathbf{x}^{(1)} + r_2 c_2 e^{r_2 t}\mathbf{x}^{(2)}$$

and

$$A\mathbf{x} = c_1 e^{r_1 t} A\mathbf{x}^{(1)} + c_2 e^{r_1 t} A\mathbf{x}^{(2)} = c_1 e^{r_1 t} r_1 \mathbf{x}^{(1)} + c_2 e^{r_1 t} r_2 \mathbf{x}^{(2)}.$$

Since $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ are not parallel they form a basis and we can find c_1 and c_2 so that

$$\mathbf{x}(0) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

In fact

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 + c_2 = a \\ ic_1 - ic_2 = b \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} c_1 = (a - ib)/2 \\ c_2 = (a + ib)/2 \end{matrix}$$

and hence

(7.6.2) $$\mathbf{x} = \frac{a - ib}{2} e^{-t-2it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{a + ib}{2} e^{-t+2it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

This is real if a,b are real, as can be seen using Euler's formulas $e^{2it} = \cos 2t + i \sin 2t$.
Sol 2 Since A is real it follows that if r is an eigenvalue with eigenvector ξ then the complex conjugate of the eigenvalue $\overline{r}$ is also an eigenvalue with complex conjugate eigenvector $\overline{\xi}$. In fact taking the complex conjugate of $A\xi = r\xi$ gives $A\overline{\xi} = \overline{r}\,\overline{\xi}$. Since $e^{rt}\xi$ is a solution it follows that the complex conjugate $e^{\overline{r}t}\overline{\xi}$ is a solution and so are the real and imaginary parts $\mathbf{u} = (e^{rt}\xi + e^{\overline{r}t}\overline{\xi})/2$ and $\mathbf{v} = (e^{rt}\xi - e^{\overline{r}t}\overline{\xi})/2i$. Writing $\xi = \mathbf{a} + i\mathbf{b}$ and $r = \lambda + i\mu$ we get after some work two real solutions

$$\mathbf{u} = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t),$$
$$\mathbf{v} = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$$

and any solution can be written as

(7.6.3) $$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}.$$

Note that in (7.6.2) $\mathbf{x} \to \mathbf{0}$, as $t \to \infty$, so $\mathbf{0}$ is a stable equilibrium. In fact this follows since the real part of the eigenvalues are negative.

7.8 Repeated eigenvalues.

Suppose now that we have a 2×2 matrix A with double eigenvalue $p(\lambda)=(\lambda-\lambda_1)^2$. We always have one eigenvector:

(7.8.1) $$(A-\lambda_1 I)\xi = 0, \qquad \xi \neq 0$$

but do we have another eigenvector that is not just a multiple of ξ? If $A=\lambda_1 I$ then any vector is an eigenvector, e.g. $(1,0)^T$, $(0,1)^T$. But if $A\neq\lambda_1 I$ then there is no other nonparallel eigenvector but instead we claim that we can find a so called **generalized eigenvector** η such that

(7.8.2) $$(A-\lambda_1 I)\eta = \xi$$

Ex 1 Find the vectors ξ and η such that (7.8.1)-(7.8.2) hold if

(7.8.3) $$A = \begin{bmatrix} \lambda_1 & d \\ 0 & \lambda_1 \end{bmatrix}, \qquad \text{where} \qquad d \neq 0,$$

Sol First we note that $p(\lambda)=(\lambda_1-\lambda)^2$. The only eigenvector is parallel to $\xi=(1,0)^T$;

(7.8.4) $$(A-\lambda_1 I)\xi = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad d\,\xi_2 = 0 \quad \Leftrightarrow \quad \xi_2 = 0$$

and e.g. $\eta=(0,1/d)^T$ satisfies (7.8.2) since

(7.8.5) $$(A-\lambda_1 I)\eta = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad d\,\eta_2 = 1 \quad \Leftrightarrow \quad \eta_2 = 1/d$$

Assume now that λ_1 is a double eigenvalue for A and that ξ and η are such that (7.8.1) and (7.8.2) hold. Then one solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}_1 = e^{\lambda_1 t}\xi.$$

We claim that

(7.8.6) $$\mathbf{x}_2 = te^{\lambda_1 t}\xi + e^{\lambda_1 t}\eta$$

is another solution. In fact, then

$$\mathbf{x}_2' = e^{\lambda_1 t}\xi + t\lambda_1 e^{\lambda_1 t}\xi + \lambda_1 e^{\lambda_1 t}\eta$$

and by (7.8.2)

$$A\mathbf{x}_2 = te^{\lambda_1 t}A\xi + \lambda_1 e^{\lambda_1 t}A\eta = te^{\lambda_1 t}\lambda_1\xi + \lambda_1 e^{\lambda_1 t}(\lambda_1\eta + \xi).$$

The general solution is therefore

$$\mathbf{x} = c_1\mathbf{x}_2 + c_2\mathbf{x}_2 = c_1 e^{\lambda_1 t}\xi + c_2\big(te^{\lambda_1 t}\xi + e^{\lambda_1 t}\eta\big).$$

Ex 2 Find the solution to the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$, where $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

Sol This is the matrix of the form (7.8.3) with $\lambda_1 = 2$ and $d = 3$ so by (7.8.4) $(1,0)^T$ is an eigenvector so one solution is $e^{2t}(1,0)^T$ and another solution is given by (7.8.6), where by (7.8.5) $\eta = (0,1/3)^T$. Hence the general solution is

$$\mathbf{x} = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\Big(te^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ 1/3 \end{bmatrix}\Big),$$

and the initial condition is satisfied if

$$\mathbf{x}(0) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1/3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad \Leftrightarrow \qquad \begin{matrix} c_1 = a \\ c_2 = 3b \end{matrix}.$$