Lecture 25: 7.7 The exponential matrix.
We will now find a nice way to the express the solution to the system

$$\mathbf{x}' = A\mathbf{x}, \tag{7.7.1}$$

where A is a 2×2 matrix, analogous to the formula for the solution of one equation. If A has two nonparallel eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ then the general solutions is

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{x}^{(1)} + c_2 e^{\lambda_2 t}\mathbf{x}^{(2)}$$

We can find two solutions to (7.7.1); $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfying the initial conditions

$$\mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using these two solutions we can express any solution

$$\mathbf{x} = a\mathbf{x}_1 + b\mathbf{x}_2 \tag{7.7.2}$$

where the constants a and b are determined by the initial condition:

$$\mathbf{x}(0) = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Let $\Phi(t)$ be the 2×2 matrix with columns the 2 vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$:

$$\Phi = [\,\mathbf{x}_1\ \mathbf{x}_2\,] = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \qquad \text{where} \qquad \mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \tag{7.7.3}$$

Then in view of the definition of matrix multiplication, (7.7.2) can be written

$$\mathbf{x}(t) = [\,\mathbf{x}_1\ \mathbf{x}_2\,]\begin{bmatrix} a \\ b \end{bmatrix} = \Phi(t)\mathbf{x}(0) \tag{7.7.4}$$

This means that when we calculated $\Phi(t)$ we can find any solution to (7.7.1) by just multiplying $\Phi(t)$ by the initial conditions $\mathbf{x}(0)$. Summarizing we have found:

Th 1 Given a 2×2 matrix A, there is a 2×2 matrix $\Phi(t)$ such that any solution of

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{7.7.5}$$

satisfies

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 \tag{7.7.6}$$

Note the analogy with the case of one equation

$$x' = ax, \qquad x(0) = x_0$$

where the solution satisfies

$$x(t) = e^{at}x_0.$$

The analogy actually goes further. Recall that we can expand in a Taylor series

$$e^{at} = 1 + ta + \frac{1}{2}t^2a^2 + \cdots + \frac{1}{k\,!}t^k a^k + \cdots$$

If A is a 2×2 matrix now define the 2×2 **exponential matrix** by

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \cdots + \frac{t^k}{k\,!}A^k + \cdots \tag{7.7.7}$$

Each term is a 2×2 matrix and one can show that each entry in the sum converges. It is not practical to use (7.7.7) but there are other ways to calculate it.
We will show that e^{At} is in fact equal to the matrix $\Phi(t)$ in Th 1. In fact,

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\Big(I + tA + \frac{t^2}{2}A^2 + \cdots + \frac{t^k}{k\,!}A^k + \cdots\Big)$$

$$= A + tA^2 + \cdots + \frac{t^{k-1}}{(k-1)\,!}A^k + \cdots = A\Big(I + tA + \cdots + \frac{t^{k-1}}{(k-1)\,!}A^{k-1}\Big) = Ae^{At}$$

and if $t = 0$

$$e^{A0} = I$$

It therefore follows that

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

satisfies (7.7.5) and $e^{At} = \Phi(t)$ in (7.7.6). In fact,

$$\frac{d}{dt}\mathbf{x} = \frac{d}{dt}e^{At}\mathbf{x}_0 == \Big(\frac{d}{dt}e^{At}\Big)\mathbf{x}_0 = Ae^{At}\mathbf{x}_0 = A\mathbf{x},$$

and

$$\mathbf{x}(0) = e^{A0}\mathbf{x}_0 = I\mathbf{x}_0 = \mathbf{x}_0$$

Ex 1 Calculate the exponential matrix for the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

Sol By Ex 7.3.2 the eigenvalues and vectors are $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and by Ex 7.5.1 the solution to $\mathbf{x}' = A\mathbf{x}$ with any initial data $\mathbf{x}(0) = \mathbf{x}_0 = (a, b)^T$ is

$$\mathbf{x} = \frac{a+b}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{a-b}{2}e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{e^{-t}+e^{3t}}{2} \\ \frac{e^{-t}-e^{3t}}{2} \end{bmatrix} a + \begin{bmatrix} \frac{e^{-t}-e^{3t}}{2} \\ \frac{e^{-t}+e^{3t}}{2} \end{bmatrix} b$$

$$= \begin{bmatrix} \frac{e^{-t}+e^{3t}}{2} & \frac{e^{-t}-e^{3t}}{2} \\ \frac{e^{-t}-e^{3t}}{2} & \frac{e^{-t}+e^{3t}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Hence the solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0, \quad \text{where} \quad \Phi(t) = \begin{bmatrix} \frac{e^{-t}+e^{3t}}{2} & \frac{e^{-t}-e^{3t}}{2} \\ \frac{e^{-t}-e^{3t}}{2} & \frac{e^{-t}+e^{3t}}{2} \end{bmatrix} = e^{At}.$$