Lecture 2: Series 11.2. If we add the terms of an infinite sequence we get a **series**:

$$a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n \qquad \left(\text{also denoted just } \sum a_n\right) \tag{11.2.1}$$

Does the sum of infinitely many terms make sense? The following series is infinity

$$1 + 1 + ... + 1 + ... = \infty$$

Ex 1 Show that the following sum approaches 1, as $n \to \infty$;

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^n} + ... = 1 \tag{11.2.2}$$

Pf In fact, suppose that you walk on the real line from 0 to 1, in smaller and smaller steps, such that the length of each step is exactly half the remaining distance to 1. The first step is a $\frac{1}{2}$ long, you end up at $\frac{1}{2}$, with $\frac{1}{2}$ remaining distance to 1.
The second step is half of the remaining interval, i.e. $\frac{1}{2}\frac{1}{2} = \frac{1}{4}$ forward, you end up at $\frac{1}{2} + \frac{1}{4}$, with $\frac{1}{4}$ remaining distance to 1. Next step is half of the preceding step. After n steps you have reached $\frac{1}{2} + \frac{1}{8} .... + \frac{1}{2^n}$, with $\frac{1}{2^n}$ remaining distance to 1. Therefore the sum of the first n terms in (11.2.2) is $1 - \frac{1}{2^n}$. But $1 - \frac{1}{2^n} \to 1$, as $n \to \infty$.

Def We say that the series (11.2.1) is convergent if the n-th **partial sums**

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$$

form a convergent sequence $\{s_n\}$, i.e. $\lim_{n\to\infty} s_n = s$ exist. In this case we write

$$\sum_{n=1}^{\infty} a_n = s = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

and we call s the **sum** of the series. The series is divergent if $\{s_n\}$ is divergent.
Ex 2 Show that the **Geometric series**

$$a + ar + ar^2 + ... + ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1}$$

diverges if $|r| \ge 1$ and converges if $|r| < 1$, with sum

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

Pf If

$$s_n = a + ar + ... + ar^{n-1}$$

then

$$rs_n = ar + ar^2 + ... + ar^{n-1} + ar^n$$

so

$$s_n - rs_n = a - ar^n$$

Hence

$$s_n = \frac{a(1-r^n)}{1-r}, \qquad r \neq 1$$

Therefore, if $|r| < 1$ then $r^n \to 0$, as $n \to \infty$, and hence

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$

If $r \le -1$ or if $r > 1$ then s_n is divergent since by the previous section r^n is divergent. If $r = 1$ then $s_n = a + a1 + ... + a1^{n-1} = na \to \infty$, as $n \to \infty$.
Ex 3 Find the sum of the series in Ex 1.
Sol It is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and hence the sum is $\frac{1}{2}/(1 - \frac{1}{2}) = 1$.

Ex 4 Show that the **harmonic series** diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ... = \infty$$

Sol If we group the terms and estimate each group from below by the number of terms times the smallest we see that the sum is bounded from below by

$$1+\frac{1}{2}+\Big(\frac{1}{3}+\frac{1}{4}\Big)+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\Big)+... \geq 1+\frac{1}{2}+2\frac{1}{4}+4\frac{1}{8}+... = 1+\frac{1}{2}+\frac{1}{2}+... = \infty$$

Th If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
The opposite of this statement is:
Divergence test If $\lim_{n\to\infty} a_n$ doesn't exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges
Note that the terms can go to 0 but the series might still diverge, see Ex 4. However, the geometric series converges exactly for those r for which the terms tend to 0.

Th A series $\sum a_n$ of positive terms either converges or diverges to infinity: $\sum a_n = \infty$
Pf $s_n = \sum_{k=1}^{n} a_k$ is increasing; $s_{n+1} = s_n + a_{n+1} \geq s_n$.
Hence by the monotonic sequence theorem it either converges or diverges to infinity.

11.3 The integral test. To determine if a series converges we can use:

The integral test Suppose f is continuous, positive, decreasing. Let $a_n = f(n)$.
Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)\, dx$ is convergent.
Ex 5 We see from this that the harmonic series in Ex 4 diverges since $\int_1^{\infty} x^{-1} dx = \infty$.

Pf of integral test: Suppose that the integral is convergent $\int_1^{\infty} f(x)\, dx < \infty$. Since f is decreasing, $f(2) \leq f(x)$ on $[1, 2]$,and hence $f(2) = \int_1^2 f(2)\, dx \leq \int_1^2 f(x)\, dx$, similarly on the interval $[n-1, n]$, $f(n) \leq f(x)$ so $f(n) \leq \int_{n-1}^{n} f(x)\, dx$. Hence

$$f(2) + ... + f(n) \leq \int_1^n f(x)\, dx \leq \int_1^{\infty} f(x)\, dx = M < \infty \qquad (11.3.1)$$

Hence $\sum a_k$ converges. Similarly, $f(n) \geq f(x)$, on $[n, n+1]$ so

$$f(1) + f(2) + ... + f(n-1) \geq \int_1^n f(x)\, dx \qquad (11.3.2)$$

If the integral tends to infinity as $n \to \infty$, then so does the left hand side.

Ex 5 For what values of p is the p-**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent.
Sol It is convergent if $p > 1$ and divergent if $p \leq 1$, as is seen by applying the test to $f(x) = x^{-p}$, since $\int_1^{\infty} x^{-p}\, dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

To estimate the remainder left over when we take finitely many terms in the series, $R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k$, we have the following estimate:

Remainder estimate with the integral test:

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_n^{\infty} f(x)\, dx.$$