Lecture 3: 11.4 The comparison test. We know some basic examples of series that converge and some others that diverge. The idea is now to determine if more series converges or diverges by comparing with the basic examples.

Ex 1 Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+3}$ converge?

Sol It looks like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Since $\displaystyle\frac{1}{2^n+3} \le \frac{1}{2^n}$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+3} \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Th A series $\sum a_n$ of positive terms either converges or diverges to infinity, i.e., it converges if and only if the partial sums are bounded; $s_n = \sum_{k=1}^n a_k \le M$, for all n.
Pf Since $s_{n+1} = s_n + a_{n+1} \ge s_n$ is increasing it follows from the monotonic convergence theorem that it converges if it is bounded.

The comparison test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n \ge N$, then $\sum a_n$ is also convergent.
(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all $n \ge N$, then $\sum a_n$ is also divergent.

Pf If (i) holds then $\displaystyle s_n = \sum_{k=1}^{n} a_k \le \sum_{k=1}^{N} a_k + \sum_{k=N-1}^{n} b_k \le C + \sum_{k=N}^{\infty} b_k = C + M < \infty$.
It therefore can not diverge to infinity so by the previous theorem it has to converge.

Ex 2 Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converge?

Sol It looks like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. We have $\displaystyle\frac{1}{2^n-1} \ge \frac{1}{2^n}$, but this is of no use to us.
However, we claim that $\displaystyle\frac{1}{2^n-1} \le \frac{2}{2^n}$, if n is sufficiently large. In fact, this would be true if $2^n \le 2(2^n-1)$, i.e. if $0 \le 2^n - 2$, which is true if $n \ge 1$.
By the comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2^n} = 2$ is convergent.

Ex 3 Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+4}$ converge?

Sol We claim that $\displaystyle\frac{1}{n+4} \ge \frac{1}{2}\frac{1}{n}$, if n is sufficiently large. In fact, this is true if $2n \ge n+4$, i.e. $n \ge 4$. Since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2}\frac{1}{n}$ is divergent it follows that the sum is divergent

The same principle holds in general:
Limit comparison test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms
If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$ exist and $0 < c < \infty$, then either both series converge or both diverge.
Pf Since $a_k/b_k \to c$, as $k \to \infty$, it follows that there is some large N such that $c/2 \le a_k/b_k \le 2c$, when $k \ge N$. Hence $a_k \le 2cb_k$ and $a_k \ge cb_k/2$ so the conditions in the comparison theorem are satisfied for $2cb_k$ respectively $cb_k/2$ in place of b_k.

11.5 The alternating series test. An **alternating series** is a series whose terms are alternately positive and negative, e.g. the **alternating harmonic series**:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...+\frac{(-1)^{n-1}}{n}+... \tag{11.5.1}$$

Is this series convergent? If we group the terms in groups of two:

$$\Big(1-\frac{1}{2}\Big)+\Big(\frac{1}{3}-\frac{1}{4}\Big)+\Big(\frac{1}{5}-\frac{1}{6}\Big)+...+\Big(\frac{1}{2n-1}-\frac{1}{2n}\Big)+... \tag{11.5.2}$$

it becomes a sum of positive terms, but we can rewrite this

$$1-\Big(\frac{1}{2}-\frac{1}{3}\Big)-\Big(\frac{1}{4}-\frac{1}{5}\Big)-\Big(\frac{1}{6}-\frac{1}{7}\Big)-...-\Big(\frac{1}{2n-2}-\frac{1}{2n-1}\Big)-\frac{1}{2n}...\le 1 \tag{11.5.3}$$

Since (11.5.2) is a series of positive terms that are bounded above it has a limit.

Similarly, ne can prove:
The alternating series test If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5+...,\qquad \text{where}\qquad b_n>0,$$

satisfies

$$0\le b_{n+1}\le b_n,\qquad \text{for all } n,\qquad \text{and}\qquad \lim_{n\to\infty}b_n=0,$$

then the series is convergent.

Alternating Series Estimate Theorem Suppose that $a_k=(-1)^{k-1}b_k$, $b_k\ge 0$, and $b_{k+1}\le b_k$. If $R_n=\sum_{k=1}^{\infty}a_k-\sum_{k=1}^{n}a_k=\sum_{k=n+1}^{\infty}a_k$, then $|\,R_n|\le b_{n+1}$.