Lecture 5: 11.8 Power series. A **power series in** $(x-a)$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ... \tag{11.8.1}$$

where x is a variable. If the sum converges it will be a function of x. In a sense it is like a polynomial of infinite degree.

Ex 1 We know that the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... + x^n + ...$$

converges for $|x| < 1$, and diverges for $|x| \geq 1$.

Ex 2 For which x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converge?

Sol Let $a_n = (x-3)^n/n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\frac{n}{(x-3)^n}\right| = \frac{n}{n+1}|x-3| = \frac{1}{1+1/n}|x-3| \to |x-3|, \quad \text{as } n \to \infty$$

By the ratio test the sum therefore converges when $|x-3| < 1$ and it diverges when $|x-3| > 1$. Now $|x-3| < 1$ is equivalent to $-1 < x-3 < 1$ which is equivalent to $2 < x < 4$. Hence the series converges for $2 < x < 4$ and diverges for $x < 2$ or $x > 4$. The ratio test gives no information when $|x-3| = 1$. If $x = 4$ the series becomes $\sum 1/n$ which is divergent since its the harmonic series. If $x = 2$ the series becomes $\sum(-1)^n/n$, which converges by the Alternating Series Test. Thus, the series converges for $2 \leq x < 4$ and diverges for other values of x.

We saw in the previous example that the the power series converged in an interval $|x-a| < R$ and divergent when $|x-a| > R$. The same result is true in general:

Th For a given power series $\sum c_n(x-a)^n$ there are only three possibilities:
(i) The series converges only when $x = a$.
(ii) The series converges for all x.
(iii) There is a positive number R such that the series is convergent if $|x-a| < R$ and divergent if $|x-a| > R$.

The number R is called the **radius of convergence**.
The theorem says nothing about the endpoints $|x-a| = R$.
In general the Ration Test (or Root Test) should be used to determine R.

Idea of proof Let $c \neq a$ be a number such that the series is convergent for $x = c$. Then $|c_n(c-a)^n| \to 0$, as $n \to \infty$ by the divergence test, so $|c_n(c-a)^n| \leq M$. Writing $|c_n(x-a)^n| \leq |c_n(c-a)^n| \left|\frac{x-a}{c-a}\right|^n \leq Mr^n$, where $r = \left|\frac{x-a}{c-a}\right|$, we see that the series converges if $r < 1$, by comparing with the geometric series.

Ex 3 What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$? **Sol** $R = \infty$ by the ratio test.

11.9 Representation of functions in terms of power series. In this section we will learn how to represent some of the standard function in terms of powers series by manipulating, differentiating or integrating the geometric series. Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + ... = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

Ex 1 Express $1/(1+x^2)$ as a power series.
Sol If we rewrite it $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$. and use the geometric series above with x replaced by $-x^2$ we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

One can also get more power series by differentiating or integrating power series:

Th If a power series $\sum c_n (x-a)^n$ has a radius of convergence $R > 0$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable when $|x-a| < R$, and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... = \sum_{n=0}^{\infty} n\, c_n (x-a)^{n-1}$$

$$\int f(x)\, dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + ... = C + \sum_{n=0}^{\infty} \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of both these power series are R.

The theorem just says that

$$\frac{d}{dx}\Big(\sum_{n=0}^{\infty} c_n (x-a)^n \Big) = \sum_{n=0}^{\infty} \frac{d}{dx}\Big(c_n (x-a)^n \Big)$$

For finite sums this is clear but the problem is convergence.

Ex Find the power series for $\ln(1-x)$ and its radius of convergence.
Sol By the previous theorem and the expansion for the geometric series:

$$-\ln(1-x) = \int \frac{dx}{1-x} = \int \big(1 + x + x^2 + ...\big)\, dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + ... = C + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Ex Find the power series for $f(x) = \tan^{-1} x$ and its radius of convergence.

Sol $\quad \tan^{-1} x = \displaystyle\int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 + ...)\, dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} + ...$