Lecture 6: Taylor series 11.10. Suppose that f can be written as a power series:
$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + ..., \qquad |x-a| < R.$
How can we then find the coefficients $c_0, c_1, ...$ from f? First, if we put $x = a$:

$$f(a) = c_0$$

By the theorem in the previous section we can differentiate the series term by term:

$$f'(x) = c_1 + 2\,c_2(x-a) + 3\,c_3(x-a)^2 + 4\,c_4(x-a)^3 + ..., \qquad |x-a| < R.$$

and substituting $x = a$ gives

$$f'(a) = c_1$$

Differentiating once more gives

$$f''(x) = 2\,c_2 + 2\cdot 3\,c_3(x-a) + 3\cdot 4\,c_4(x-a)^2 + ..., \qquad |x-a| < R.$$

and substituting $x = a$ gives

$$f''(a) = 2\,c_2$$

Differentiating once more gives

$$f'''(x) = 2\cdot 3\,c_3 + 2\cdot 3\cdot 4\,c_4(x-a) + ..., \qquad |x-a| < R.$$

and substituting $x = a$ gives

$$f'''(a) = 2\cdot 3\,c_3$$

If you continue to differentiate and substitute you will get

$$f^{(n)}(a) = 2\cdot 3\cdot 4\cdot\dots\cdot n, c_n = n!\,c_n$$

Hence, if $f(x)$ has a power series expansion at a; then the coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

In other words, if f has a power series expansion at a then
(11.10.1)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3\cdot 2}(x-a)^3 + ...$$

The series above is called the **Taylor series of the function f at a**. For the special case of $a = 0$ it is called the **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(a) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3\cdot 2}x^3 + ...$$

We remark that one has to prove that a function is equal to its Taylor series even if it converges. In fact this is not always true as is seen by problem 62 in the book.

Ex Find the Maclaurin series for $f(x) = e^x$ and its radius of convergence.
Sol $f'(x) = e^x$,..., $f^{(n)}(x) = e^x$, and $f^{(n)}(0) = e^0 = 1$, so the Maclaurin series is

(11.10.2)
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2} + \cdots$$

To find the radius of convergence let $a_n = x^n/n!$. Then as $n \to \infty$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1,$$

so the series converges for all x by the Ratio Test. The radius of convergence $R = \infty$.
The conclusion is that if e^x have a power series the it is (11.10.2).

When is a function, that have derivatives of all orders, equal to its Taylor series (11.10.2)? Let the **n-th degree Taylor polynomial of f at a** be given by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k\,!}(x-a)^n = f(a)+f\,'(a)(x-a)+\frac{f\,''(a)}{2}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n\,!}(x-a)^n$$

Notice that $T_n(x)$ is the polynomial of degree n that best approximates f close to a.
Ex Draw the graph of the exponential function and its first Taylor polynomials.
Sol $T_1(x) = 1 + x$ is the tangent line to the exponential function at $x = 0$, and $T_2(x)=1+x+x^2/2$ also has the same curvature as the exponential function at $x=0$.

In general f is the sum of its Taylor series if the **remainder**

$$R_n(x) = f(x) - T_n(x) \to 0,$$

tends to 0; $R_n(x) \to 0$, as $n \to \infty$. In order to prove this we can use:

Th(Taylor's Inequality) If $|f^{(n+1)}(x)| \le M$, for $|\,x - a| \le d$, then

$$(11.10.3) \qquad |R_n(x)| \le \frac{M}{(n+1)\,!}|\,x - a|^{n+1}, \qquad \text{for } |\,x - a| \le d.$$

Ex Prove that $f(x) = e^x$ is equal to its Maclaurin series for all x.
Sol Since $f^{(n)}(x) = e^x$, it follows that $|f^{(n)}(x)| \le e^d = M$, when $|x| \le d$. Hence by Taylor's inequality

$$|R_n(x)| \le \frac{e^d}{(n+1)\,!}|x|^{n+1}, \qquad \text{for } |x| \le d.$$

Since $|x|^n/n! \to 0$, when $n \to \infty$, it follows that $R_n(x) \to 0$, when $n \to \infty$, for any x.

Pf of (11.10.3) for $n = 1$. If $f\,''(x) \le M$ then for $a \le x \le a + d$:

$$f\,'(x) - f\,'(a) = \int_a^x f\,''(t)\,dt \le \int_a^x M\,dt = M(x - a)$$

where we used the Fundamental Theorem of Calculus, f' is an antiderivative of f''. Using the inequality that we just derived

$$f\,'(x) \le f\,'(a) + M(x - a), \qquad a \le x \le a + d$$

we similarly obtain

$$f(x) - f(a) = \int_a^x f\,'(t)\,dt \le \int_a^x \big(f\,'(a) + M(t - a)\big)\,dt = f\,'(a)(x - a) + M\frac{(x - a)^2}{2}$$

With $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f\,'(a)(x - a)$ we have shown that

$$R_1(x) \le \frac{M}{2}(x - a)^2$$

A similar argument using that $f\,''(x) \ge -M$ shows the inequality in the other direction and we get

$$|R_1(x)| \le \frac{M}{2}(x - a)^2$$

Although the calculations where for $x > a$, similar calculations shows it for $x < a$.