

Lecture 7: 11.10, 12 Application of Taylor series. Recall that the n -th degree Taylor polynomial of f at a is given by

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

If $n = \infty$ it is called the Taylor series of f at a . By Taylor's theorem

$$f(x) = T_n(x) + R_n(x),$$

where the remainder satisfies Taylor's inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}, \quad \text{for } |x-a| \leq d,$$

where M is a number such that $|f^{(n+1)}(x)| \leq M$, for $|x-a| \leq d$.

Ex Find the Maclaurin series for $f(x) = \cos x$. Show that it is equal to $\cos x$ for all x .

Sol Differentiating we get

$$f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x,$$

after which it repeats itself, so

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1,$$

Hence the Maclaurin series is

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Since $f^{(n+1)}(x)$ is either $\pm \cos x$ or $\pm \sin x$ it follows that $|f^{(n+1)}(x)| \leq 1$. Hence with $M = 1$ in Taylor's inequality:

$$(11.1.4) \quad |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

Since this tends to 0 as $n \rightarrow \infty$ it follows that $f(x)$ is equal to its Maclaurin series.

Ex Find $\cos 3^\circ$ to within five decimal places.

Sol $3^\circ = 3(2\pi/360) = \pi/60$ radians. The error in using a Taylor polynomial of degree n is by (11.1.4) less than 0.000005 if

$$|R_n(x)| \leq \frac{(\pi/60)^{n+1}}{(n+1)!} \leq \frac{10^{-n-1}}{(n+1)!} < 0.000005$$

If we pick $n = 3$ we get

$$\frac{(\pi/60)^4}{4!} \leq \frac{10^{-4}}{24} \leq \frac{10^{-5}}{2} = 0.000005$$

Using the Maclaurin polynomial of degree 3 we get

$$(11.10.5) \quad \cos(\pi/60) \sim T_3(\pi/60) = 1 - \frac{(\pi/60)^2}{2}$$

Alternatively since it is an alternating series we know that the error in (11.10.5) is bounded by the next term left out in the series, which is exactly the same as above.

Ex Approximate $f(x) = \sqrt{x}$ by a Taylor polynomial of degree 2 at $a = 25$. Use it to find an approximation for $\sqrt{24}$. Show that your error is less than 0.0001.

Sol We have

$$f(x) = x^{1/2}, \quad f'(x) = \frac{1}{2} \frac{1}{x^{1/2}}, \quad f''(x) = -\frac{1}{2} \frac{1}{2} \frac{1}{x^{3/2}}, \quad f'''(x) = \frac{1}{2} \frac{1}{2} \frac{3}{2} \frac{1}{x^{5/2}}$$

and hence since $\sqrt{25} = 5$

$$f(25) = 5, \quad f'(25) = \frac{1}{2} \frac{1}{5} = \frac{1}{10}, \quad f''(25) = -\frac{1}{2} \frac{1}{2} \frac{1}{25\sqrt{25}} = -\frac{1}{500},$$

Hence

$$\sqrt{x} \sim 5 + \frac{x-25}{10} - \frac{(x-25)^2}{1000}$$

and so

$$\sqrt{24} \sim 5 - \frac{1}{10} - \frac{1}{1000} = 4.899$$

Since $f'''(x)$ is decreasing when $x \geq 0$ it follows that

$$|f'''(x)| = \frac{3}{8} \frac{1}{x^{5/2}} \leq \frac{3}{8} \frac{1}{24^{5/2}}, \quad \text{when } 24 \leq x \leq 25,$$

But

$$\frac{3}{8} \frac{1}{24^{5/2}} = \frac{3}{8} \frac{1}{24 \cdot 24 \cdot \sqrt{24}} = \frac{1}{64 \cdot 24\sqrt{24}} \leq \frac{1}{64 \cdot 24\sqrt{16}} = \frac{1}{64 \cdot 24 \cdot 4} = M$$

Hence by Taylor's inequality the error is bounded by

$$|R_2(x)| \leq \frac{1}{64 \cdot 24 \cdot 4} \frac{|x-25|^3}{6} = \frac{|x-25|^3}{4096 \cdot 9} = \frac{|x-25|^3}{36864}, \quad |x-25| \leq 1$$

Hence

$$|R_2(24)| \leq \frac{1}{36864} \leq 110000 = 0.0001$$

Note that in this case we can not estimate the error by the next term left out since the terms are not alternating.

Ex Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{e^x - 1 - x}$.

Sol Expanding in Taylor series we see that the limit is

$$\lim_{x \rightarrow 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{4!} \dots - 1}{1 + x + \frac{x^2}{2} + \dots - 1 - x} = \lim_{x \rightarrow 0} \frac{-\frac{x^2}{2} + \frac{x^3}{3!} + \dots}{\frac{x^2}{2} + \dots} = \lim_{x \rightarrow 0} \frac{-\frac{1}{2} + \frac{x^2}{4!} + \dots}{\frac{1}{2} + \frac{x}{3!}} = -1$$