Lecture 8: 1.1-2. Two Models, Direction fields, Solution curves.

Model I: A Falling object is subject to Newton's second law:

$$F = m\,a \qquad \text{Force} = \text{mass} \times \text{acceleration}$$

where

$$a = \frac{dv}{dt} \qquad \text{acceleration} = \text{the time derivative of the velocity}$$

and

$$F = mg - \gamma v \qquad \text{Force} = \text{gravitational force} - \text{air resistance friction}$$

where $g \sim 9.8$ is the gravitational constant and $\gamma > 0$ is due to air resistance. Hence

$$m\frac{dv}{dt} = mg - \gamma v$$

If we plug in certain values of the constants we get

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \equiv f(v) \tag{1.1.1}$$

where the last equality defines $f(v)$.

Given v, $f(v)$ in (1.1) gives the slope dv/dt, ie. the rate of increase of the velocity. We can display this information graphically in the t-v plane by drawing small arrows with slope $f(v)$, i.e. vectors in the direction $(1, f(v))$, at a number of points (t, v) in the t-v plane. This is called a **direction field** or **slope field**.

One can the approximately calculate the **solution curve** starting at some point (t, v) in the t-v plane by going in the direction of the arrow $f(v)$ in the t-v plane for some small time interval, and then at the new point going in the direction of the arrow at that point for some small time interval, and so on.

E.g. if $v = 60$ then $f(v) = f(60) = -2.2$, independently of what t is, and $f(40) = 1.8$ and $f(50) = -0.2$, see Figure 1.1.2 in the text book.

Notice $f(v)$ changes sign, if $f(60) < 0$ and $f(40) > 0$, and in fact $f(49) = 0$. This means that if $v = 49$ then $dv/dt = 0$ so v stays constant.

A point where $f(v) = 0$ is called an **equilibrium** point. In this case it is in fact a **stable equilibrium** because if $v > 49$ then $f(v) < 0$ so $dv/dt < 0$ and it will push v down towards 49, and if $v < 49$ then $f(v) > 0$ so $dv/dt > 0$ which will push v up towards 49. It is called stable because any points in the t-v plane with v close to 49 will be pulled in towards $v = 49$, in fact graphically it looks like eventually all paths will approach $v = 49$ as time $t \to \infty$.

Model II: Mice and Owls. Consider a population p of field mice who inhabit a certain area. In the absence of predators it is natural to assume that the population increases at a rate proportional to the current population. The proportionality factor r is called the **growth rate**. Now, let us also suppose that several owls live in the same area and eats a fixed number of mice every day. In some units we have

$$\frac{dp}{dt} = 0.5p - 450 = g(v) \tag{1.1.2}$$

If, we as for the earlier model draw the direction field we notice a big difference. There is an equilibrium at $p = 900$, for which $g(p) = g(900) = 0$, but for $p > 900$, $g(p)$ so the population increases and hence move away from 900. Similarly, if $p < 900$ then $g(p) < 0$ so the population decreases and hence also move away from $p = 900$. We conclude that this is an **unstable equilibrium**

Section 1.2 Analytical solution of the simple models. We can actually solve the equation (1.1.2) analytically:

$$\frac{dp/dt}{p - 900} = 2$$

and hence by the chain rule for derivatives, this is

$$\frac{d}{dt} \ln |p - 900| = \frac{1}{2}$$

which is equivalent to

$$\ln |p - 900| = \frac{t}{2} + C$$

or after integration:

$$|p - 900| = e^C e^{t/2}$$

i.e.

$$p - 900 = \pm e^C e^{t/2} = ce^{t/2}$$

where $c = \pm e^C$. The constant c will be determined by initial conditions: If $p(0) = 850$, then $-50 = p - 900 = ce^0$, so $p = 950 - 50e^{t/2}$ which is indeed decreasing fast towards 0 Similarly if $p(0) = 950$, then $50 = p - 900 = ce^0$ so $p = 900 + 50e^{t/2}$, which is indeed increasing fast towards infinity.

Note that there is a similar calculation for the first model.