

Formulas for Midterm 1.

Space curve C : $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$.

Tangent vector: $\mathbf{R}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Arc length: $L = \int_a^b |\mathbf{R}'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$

Gradient: $\nabla f(x, y, z) = \mathbf{grad} f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial f}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial f}{\partial z}(x, y, z)\mathbf{k}$

Chain rule: If $w(t) = f(\mathbf{R}(t))$ then $\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = \nabla f \cdot \frac{d\mathbf{R}}{dt}$

Directional derivative: $\left. \frac{df}{ds} = \frac{d}{ds} f(\mathbf{R}(s)) \right|_{s=0} = \nabla f \cdot \mathbf{u}$, if $\mathbf{R}(s) = \mathbf{R}_0 + s\mathbf{u}$, $|\mathbf{u}| = 1$.

Vector Field: $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$.

Flow line: $\frac{d\mathbf{R}(t)}{dt} = \beta \mathbf{F}(\mathbf{R}(t))$, $\beta \geq 0$ i.e.
$$\begin{cases} x' = \beta F_1 \\ y' = \beta F_2 \\ z' = \beta F_3 \end{cases} \quad \text{or} \quad \frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3}.$$

Divergence: $\nabla \cdot \mathbf{F} = \text{div } \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

Curl: $\nabla \times \mathbf{F} = \mathbf{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \mathbf{k}$,

where $\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$ etc.

Line Integral: $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t) dt = \int_C F_1 dx + F_2 dy + F_3 dz$
 $= \int_a^b \left(F_1(x(t), y(t), z(t)) \frac{dx}{dt} + F_2(x(t), y(t), z(t)) \frac{dy}{dt} + F_3(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$

Conservative: $\mathbf{F} = \mathbf{grad } \phi$, i.e.
$$\begin{cases} \partial\phi/\partial x = F_1 \\ \partial\phi/\partial y = F_2 \\ \partial\phi/\partial z = F_3 \end{cases}$$

Irrotational: $\nabla \times \mathbf{F} = 0$.

Th 1: If \mathbf{F} is continuously differentiable in a domain D then \mathbf{F} is conservative if and only if the line integral along any curve C in D only depends on the endpoints. In that case $\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(P) - \phi(Q)$, where Q is the initial and P is the endpoint.

Th 2: If \mathbf{F} is continuously differentiable in a simply connected domain D then \mathbf{F} is conservative if and only if it is irrotational.