Lecture 10b: Section 4.5: Solenoidal vector fields. We show the following:

Theorem 1. *Suppose that* $\mathbf{F}$ *is continuously differentiable in a star-shaped domain. Then* $\mathbf{F}$ *is irrotational (i.e.* $\mathbf{curl}\,\mathbf{F} = 0$*) if and only if it is conservative (i.e. there is scalar potential* ϕ *such that* $\mathbf{F} = \mathbf{grad}\ \phi$*.)*

Theorem 2. *Suppose that* $\mathbf{F}$ *is continuously differentiable in a star-shaped domain. Then* $\mathbf{F}$ *is solenoidal (i.e.* $\operatorname{div}\mathbf{F} = 0$*) if and only if there is a vector potential* $\mathbf{G}$*. such that* $\mathbf{F} = \mathbf{curl}\,\mathbf{G}$*.*

In section 5.3 it is shown that one can write any vector field as a sum

$$\mathbf{F} = \mathbf{H} + \mathbf{J}, \qquad \text{where} \qquad \mathbf{curl}\,\mathbf{H} = 0, \quad \text{and} \quad \operatorname{div}\mathbf{J} = 0. \tag{4.5.1}$$

(In fact, in section 5.2 it will be shown that we can solve Laplace equation $\nabla\cdot\nabla\phi = \triangle\phi = \nabla\cdot\mathbf{F}$. Then $\nabla\cdot\big(\mathbf{F} - \nabla\phi\big) = 0$ so we can take $\mathbf{H} = \nabla\phi$ and $\mathbf{J} = \mathbf{F} - \mathbf{H}$.) Then by Theorem 1, $\mathbf{H} = \mathbf{grad}\ \phi$ and by Theorem 2, $\mathbf{J} = \mathbf{curl}\,\mathbf{G}$.

In section 4.4 we stated Theorem 1 for simply connected domains but only proved it for a box. A star-shaped domain D is defined by that there is a fixed point $\mathbf{R}_0 \in D$ such that for every point $\mathbf{R} \in D$ the line segment from $\mathbf{R}_0$ to $\mathbf{R}$ is contained in D, i.e. $\mathbf{r}(t) \in D$. Let use assume that $\mathbf{R}_0 = \mathbf{0}$. Then for $0 \le t \le 1$, $\mathbf{r}(t) \in D$, where

$$\mathbf{r}(t) = t\mathbf{R}, \quad \text{where} \quad \mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \tag{4.5.2}$$

To prove Theorem 1 we define

$$\phi(\mathbf{R}) = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r})\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^1 F_1(\mathbf{r})x + F_2(\mathbf{r})y + F_3(\mathbf{r})z\,dt \tag{4.5.3}$$

Then for example

$$\frac{\partial}{\partial x}F_3(\mathbf{r}) = \frac{\partial}{\partial x}F_3\big(tx, ty, tz\big) = \frac{\partial F_3}{\partial x}(\mathbf{r})\,t \tag{4.5.4}$$

Note that the left means the derivative of the composite function $F_3(\mathbf{r})$, where $\mathbf{r}$ depends on x through (4.5.2), and the right is the derivative of the function $F_3(x, y, z)$ evaluated at $(x, y, z) = \mathbf{r}$. Since we assumed that $\partial F_3/\partial x = \partial F_1/\partial z$;

$$\frac{\partial}{\partial x}F_3(\mathbf{r}) = \frac{\partial F_1}{\partial z}(\mathbf{r})\,t \tag{4.5.5}$$

It follows that

$$\frac{\partial\phi}{\partial x}(\mathbf{R}) = \int_0^1 F_1(\mathbf{r}) + t\Big(x\frac{\partial F_1}{\partial x} + y\frac{\partial F_1}{\partial y} + z\frac{\partial F_1}{\partial z}\Big)(\mathbf{r})\,dt \tag{4.5.6}$$

Using the chain rule:

$$\frac{d}{dt}F_1(\mathbf{r}) = \frac{d}{dt}F_1\big(tx, ty, tz\big) = \Big(\frac{\partial F_1}{\partial x}x + \frac{\partial F_1}{\partial y}y + \frac{\partial F_1}{\partial z}z\Big)(\mathbf{r}) \tag{4.5.7}$$

we have

$$\frac{\partial\phi}{\partial x}(\mathbf{R}) = \int_0^1 F_1(\mathbf{r}) + t\frac{d}{dt}F_1(\mathbf{r})\,dt = \int_0^t \frac{d}{dt}\Big(tF_1(\mathbf{r})\Big)\,dt = F_1(\mathbf{r}(1)) = F_1(\mathbf{R}) \tag{4.5.8}$$

This shows that $\partial\phi/\partial x = F_1$. In the same way $\partial\phi/\partial y = F_1$ and $\partial\phi/\partial z = F_3$.

To prove Theorem 2 we define

$$\mathbf{G}(\mathbf{R}) = \int_0^1 t\,\mathbf{F}(\mathbf{r}) \times \frac{d\mathbf{r}}{dt}\,dt, \tag{4.5.9}$$

where $\mathbf{r} = t\mathbf{R}$ is given by (4.5.2). Then

$$\begin{aligned} \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= \int_0^1 t\,\frac{\partial}{\partial x}\Big(xF_3(\mathbf{r}) - zF_1(\mathbf{r})\Big) - t\,\frac{\partial}{\partial y}\Big(zF_2(\mathbf{r}) - yF_3(\mathbf{r})\Big)\,dt \\ &= \int_0^1 t\,\Big(x\frac{\partial}{\partial x}F_3(\mathbf{r}) - z\frac{\partial}{\partial x}F_1(\mathbf{r})\Big) - t\,\Big(z\frac{\partial}{\partial y}F_2(\mathbf{r}) - y\frac{\partial}{\partial y}F_3(\mathbf{r})\Big) + 2\,tF_3(\mathbf{r})\,dt \\ &= \int_0^1 t^2\,\Big(x\frac{\partial F_3}{\partial x}(\mathbf{r}) - z\frac{\partial F_1}{\partial x}(\mathbf{r})\Big) - t^2\,\Big(z\frac{\partial F_2}{\partial y}(\mathbf{r}) - y\frac{\partial F_3}{\partial y}(\mathbf{r})\Big) + 2\,tF_3(\mathbf{r})\,dt \\ &= \int_0^1 t^2\,\Big(x\frac{\partial F_3}{\partial x} + y\frac{\partial F_3}{\partial y} + z\frac{\partial F_3}{\partial z}\Big)(\mathbf{r}) - t^2 z\Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big)(\mathbf{r}) + 2\,tF_3(\mathbf{r})\,dt, \end{aligned} \tag{4.5.10}$$

Here the last term vanishes since we assumed that $\operatorname{div}\mathbf{F} = 0$ and the first term is again by the chain rule

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \int_0^t t^2\frac{d}{dt}F_3(\mathbf{r}) + 2\,tF_3(\mathbf{r})\,dt = \int_0^t \frac{d}{dt}\Big(t^2F_3(\mathbf{r})\Big)\,dt = F_3(\mathbf{r}(1)) = F_3(\mathbf{R}) \tag{4.5.11}$$

Similarly, it follows that $\partial G_2/\partial z - \partial G_3/\partial y = F_1$ and $\partial G_3/\partial x - \partial G_1/\partial z = F_2$.
Ex. Let $\mathbf{F} = \mathbf{A}$, where $\mathbf{A}$ is a constant vector. Show that $\operatorname{div}\mathbf{F} = 0$ and find a vector potential $\mathbf{F} = \mathbf{curl}\,\mathbf{G}$.
Sol. $\operatorname{div}\mathbf{A} = 0$ since $\mathbf{A}$ is constant. Using the formula (4.5.9) with $\mathbf{R}_0 = \mathbf{0}$ we get

$$\mathbf{G}(\mathbf{R}) = \int_0^1 t\,\mathbf{F}(t\mathbf{R}) \times \mathbf{R}\,dt = \int_0^1 t\,\mathbf{A} \times \mathbf{R}\,dt = \mathbf{A} \times \mathbf{R}\int_0^1 t\,dt = \frac{1}{2}\mathbf{A} \times \mathbf{R}, \tag{4.5.12}$$

Ex. Let $\mathbf{F} = x\mathbf{j}$. Show that $\operatorname{div}\mathbf{F} = 0$ and find a vector potential $\mathbf{F} = \mathbf{curl}\,\mathbf{G}$.
Sol Instead of using the formula try to solve

$$\mathbf{curl}\,\mathbf{G} = \Big(\frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y}\Big)\mathbf{i} + \Big(\frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial G_1}{\partial y} - \frac{\partial G_2}{\partial x}\Big)\mathbf{k} = x\,\mathbf{j} \tag{4.5.13}$$

i.e.

$$\frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y} = 0, \qquad \frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z} = x, \qquad \frac{\partial G_1}{\partial y} - \frac{\partial G_2}{\partial x} = 0 \tag{4.5.14}$$

Just try with $G_2 = 0$ and $G_1 = G_1(x,x)$ and $G_3 = G_3(x,z)$ independent of y. Then the first and the last equation are fulfilled. To solve the second equation we can also take $G_1 = 0$ in which case the second equation become $\partial G_3/\partial x = x$ which has the solution $G_3 = x^2/2$ for example. Hence $\mathbf{G} = \mathbf{k}\,x^2/2$ will do.