Lecture 10: Section 4.4 Conservative fields-Irrotational fields. We have seen in the last section an example of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ that could not possibly be conservative because if there was a potential such that $\mathbf{F} = \mathbf{grad}\ \phi$ then because $\partial^2\phi/\partial x\partial y = \partial^2\phi\partial y\partial x$ etc. we must have $\partial F_1/\partial y = \partial F_2/\partial x$ etc. Hence for a vector field to be conservative we must have

$$(4.4.1)\qquad \mathbf{curl}\,\mathbf{F} = \Big(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\Big)\mathbf{i} + \Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\Big)\mathbf{k} = \mathbf{0}$$

A vector field satisfying this is called **irrotational**. We have

Theorem. *A vector field* $\mathbf{F}$ *defined and continuously differentiable throughout a simply connected domain* D *is conservative if and only if it is irrotational in* D.

The physical interpretation of this is that the flow lines for a gradient vector field can not curl around in a closed orbit since ϕ increases in the direction of the gradient.
Ex. 4.4.1 Show that $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ is conservative in all space.
Sol. By the previous theorem it suffices to show that it is irrotational: $\mathbf{curl}\,\mathbf{F}=\mathbf{0}$.
Ex. 4.4.2 Is $\mathbf{F} = (-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)$ conservative in $D=\{(x,y,z);\ (x,y)\neq(0,0)\}$.
Sol. $\nabla\times\mathbf{F} = ... = \mathbf{0}$ but D is not simply connected so it does not follow that it is conservative. In fact, it is not since the line line integral along a circle around the z-axis is nonvanishing as we shall see. If $x = \cos t$, $y = \sin t$ and $z = 0$, $0 \leq t \leq 2\pi$ then $\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi}(F_1 dx/dt + F_2 dy/dt + F_3 dz/dt)dt = \int_0^{2\pi}\sin^2 t + \cos^2 t\, dt = 2\pi$.
Ex. 4.4.3 Show that the vector field in Example 4.4.2 in the domain D of space where $x > 0$ is conservative and find a potential there.
Sol. That it is conservative follows from that $\nabla\times\mathbf{F} = \mathbf{0}$ and that the domain D is simply connected. $\phi = \tan^{-1}(y/x)$ is a potential in D, since $x \neq 0$ in D.

Proof of the theorem. That conservative implies irrotational is just the calculation above that $\nabla\times\nabla\phi = 0$. We shall prove that irrotational implies conservative if the domain is all of space or a rectangular box containing the origin. We define

$$(4.4.2)\qquad \phi(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

and we will show that $\nabla\phi=\mathbf{F}$ if $\nabla\times\mathbf{F}=0$. By the Fundamental Theorem of Calculus

$$(4.4.3)\qquad \frac{\partial\phi}{\partial z}(x,y,z) = \frac{d}{dz}\Big|_{(x,y)-fixed}\phi(x,y,z) = \frac{d}{dz}\int_0^z F_3(x,y,t)\,dt = F_3(x,y,z)$$

Similarly, also using that we can differentiate below an integral sign:

$$(4.4.4)\qquad \frac{\partial\phi}{\partial y}(x,y,z) = F_2(x,y,0) + \int_0^z \frac{\partial F_3}{\partial y}(x,y,t)\,dt$$

If we now also use the fact that $\partial F_3/\partial y = \partial F_2/\partial z$ we obtain

$$(4.4.5)\qquad \frac{\partial\phi}{\partial y}(x,y,z) = F_2(x,y,0) + \int_0^z \frac{dt}{dt}\Big|_{(x,y)-fixed}F_2(x,y,t)\,dt$$
$$= F_2(x,y,0) + F_2(x,y,z) - F_2(x,y,0) = F_2(x,y,z)$$

by the Fundamental Theorem of Calculus. Finally by the same argument

$$(4.4.4)\qquad \frac{\partial\phi}{\partial x}(x,y,z) = F_1(x,0,0) + \int_0^y \frac{\partial F_2}{\partial x}(x,t,0)\,dt + \int_0^z \frac{\partial F_3}{\partial x}(x,y,t)\,dt$$
$$= F_1(x,0,0) + \int_0^y \frac{d}{dt}F_1(x,t,0)\,dt + \int_0^z \frac{d}{dt}F_1(x,y,t)\,dt$$
$$= F_1(x,0,0) + F_1(x,y,0) - F_1(x,0,0) + F_1(x,y,z) - F_1(x,y,0) = F_1(x,y,z)$$