

Lecture 11: Section 3.7: Taylor series. Recall Taylor's formula for $f: \mathbf{R} \rightarrow \mathbf{R}$:

$$(3.7.1) \quad f(x) = f(a) + f'(a)(x-a) + \frac{f''}{2}(a)(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x, a)$$

where the remainder or error tends to 0 faster than the previous terms when $x \rightarrow a$:

$$(3.7.2) \quad |R_k(x, a)| \leq \frac{M}{(k+1)!} |x-a|^{k+1}, \quad \text{if } |f^{(k+1)}(z)| \leq M,$$

for $|z-a| \leq |x-a|$. We refer to the textbook for Math 21D for a proof of this.

A similar formula holds for functions of several variables $\mathbf{F}: \mathbf{R}^n \rightarrow \mathbf{R}^m$. In order to state it we first write $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{R}^n$.

$$(3.7.3) \quad \mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{a}) + \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^n \mathbf{F}_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j) \\ + \dots + \frac{1}{k!} \sum_{i_1, \dots, i_k=1}^n \mathbf{F}_{x_{i_1} \dots x_{i_k}}(\mathbf{a})(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k}) + \mathbf{R}_k(\mathbf{a}, \mathbf{x}),$$

where the remainder or error tends to 0 faster than the previous terms when $x \rightarrow a$:

$$(3.7.4) \quad |\mathbf{R}_k(x, a)| \leq \frac{M}{(k+1)!} |\mathbf{x} - \mathbf{a}|^{k+1}, \quad \text{if } \sum_{i_1, \dots, i_{k+1}=1}^n |\mathbf{F}_{x_{i_1} \dots x_{i_{k+1}}}(\mathbf{z})| \leq M,$$

for $|\mathbf{z} - \mathbf{a}| \leq |\mathbf{x} - \mathbf{a}|$. Here

$$(3.7.5) \quad \mathbf{F}_{x_{i_1} \dots x_{i_k}} = \frac{\partial^k \mathbf{F}}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

First, the general case reduces to the case $m = 1$ by considering each component of $\mathbf{F} = (F_1, \dots, F_m)$ and we may hence assume that $\mathbf{F}: \mathbf{R}^n \rightarrow \mathbf{R}$. In order to prove (3.7.3) we introduce $\mathbf{x} - \mathbf{a} = \mathbf{h}$ and apply the one dimensional Taylor's formula (3.7.1) to the function $f(t) = \mathbf{F}(\mathbf{x}(t))$ along the line segment $\mathbf{x}(t) = \mathbf{a} + t\mathbf{h}$, $0 \leq t \leq 1$:

$$(3.7.6) \quad f(1) = f(0) + f'(0) + f''(0)/2 + \dots + f^{(k)}(0)/k! + R_k$$

Here $f(1) = \mathbf{F}(\mathbf{a} + \mathbf{h})$, i.e. the left hand side of (3.7.3), $f(0) = \mathbf{F}(\mathbf{a})$, i.e. the first term in the right hand side of (3.7.3), and by the chain rule

$$(3.7.7) \quad f'(t) = \frac{d}{dt} \mathbf{F}(\mathbf{x}(t)) = \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{x}(t)) \frac{dx_i}{dt} = \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{x}(t)) h_i$$

and hence $f'(0)$ is the second term in the right of (3.7.3). Repeating this gives

$$(3.7.8) \quad f''(t) = \frac{d}{dt} \sum_{i=1}^n \mathbf{F}_{x_i}(\mathbf{x}(t)) h_i = \sum_{i=1, j=1}^n \mathbf{F}_{x_i x_j}(\mathbf{x}(t)) h_i h_j$$

and this gives the third term and so on.

Ex. Let $f(x, y) = 3 + 2x + x^2 + 2xy + 3y^2 + x^3 - y^4$. Find the second degree Taylor polynomial around $\mathbf{a} = (0, 0)$. **Sol.** The second degree Taylor polynomial is

$$(3.7.9) \quad f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} (f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2) \\ = 3 + 2x + \frac{1}{2} (2x^2 + 2 \cdot 2xy + 6y^2) = 3 + 2x + x^2 + 2xy + 3y^2$$

The derivative of a vector field as a linear map. Let $\mathbf{F} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a vector field. Then we can think of the derivative of \mathbf{F} at the point $\mathbf{a} \in \mathbf{R}^n$ as the linear map $\mathbf{DF} : \mathbf{R}^n \rightarrow \mathbf{R}^n$, mapping the vector $\mathbf{h} = (h_1, \dots, h_n)$ to the vector

$$(3.7.10) \quad \mathbf{DF}(\mathbf{a})\mathbf{h} = \lim_{t \rightarrow 0} \frac{\mathbf{F}(\mathbf{a} + t\mathbf{h}) - \mathbf{F}(\mathbf{a})}{t} = \frac{\partial \mathbf{F}}{\partial x_1}(\mathbf{a})h_1 + \dots + \frac{\partial \mathbf{F}}{\partial x_n}(\mathbf{a})h_n,$$

This is the linear map that best approximates the function close to \mathbf{a} :

$$(3.7.11) \quad \mathbf{F}(\mathbf{a} + \mathbf{h}) = \mathbf{F}(\mathbf{a}) + \mathbf{DF}(\mathbf{a})\mathbf{h} + \mathbf{R}_2(\mathbf{a}, \mathbf{h}), \quad \text{where} \quad |\mathbf{R}_2(\mathbf{a}, \mathbf{h})| \leq M|\mathbf{h}|^2$$

tends to 0 faster than the other terms as $|\mathbf{h}| \rightarrow 0$.

Ex. Let $\mathbf{F}(x, y) = (1 + x - y + x^2)\mathbf{i} + (x^2 - y^2 + y^4)\mathbf{k}$. Find the Taylor polynomial of degree one for $\mathbf{F}(x, y)$ around $(x, y) = (1, 0)$. **Sol.** We have

$$(3.7.12) \quad \frac{\partial \mathbf{F}}{\partial x}(x, y) = ((1 + 2x)\mathbf{i} + 2x\mathbf{j}), \quad \frac{\partial \mathbf{F}}{\partial y}(x, y) = (-\mathbf{i} + (-2y + 4y^3)\mathbf{j})$$

Hence

$$(3.7.13) \quad \mathbf{DF}(1, 0)(x - 1, y) = \frac{\partial \mathbf{F}}{\partial x}(1, 0)(x - 1) + \frac{\partial \mathbf{F}}{\partial y}(1, 0)y = (3\mathbf{i} + 2\mathbf{j})(x - 1) - \mathbf{i}y$$

so

$$(3.7.14) \quad \begin{aligned} \mathbf{F}(x, y) &= \mathbf{F}(1, 0) + \mathbf{DF}(1, 0)(x - 1, y) + \mathbf{R}_2(1, 0)(x - 1, y) \\ &= 3\mathbf{i} + \mathbf{j} + (3\mathbf{i} + 2\mathbf{j})(x - 1) - \mathbf{i}y + \mathbf{R}(1, 0)(x - 1, y) \\ &= (3x - y)\mathbf{i} + (2x - 1)\mathbf{j} - \mathbf{R}_2(1, 0)(x - 1, y) \end{aligned}$$

In fact, the remainder

$$(3.7.15) \quad \begin{aligned} \mathbf{R}_2(1, 0)(x - 1, y) &= (1 + x - y + x^2)\mathbf{i} + (x^2 - y^2 + y^4)\mathbf{k} - ((3x - y)\mathbf{i} + (2x - 1)\mathbf{j}) \\ &= (1 + x^2 - 2x)\mathbf{i} + (x^2 - 2x + 1 + y^2 + y^4)\mathbf{j} = (x - 1)^2\mathbf{i} + ((x - 1)^2 + y^2 + y^4)\mathbf{j} \end{aligned}$$

is at least quadratically small in $(x - 1)$ and y , i.e. it is bounded by a constant times $(x - 1)^2 + y^2$, when $(x - 1)^2 + y^2$ is small.

Let us now consider the special case of a vector field $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$. Then the total derivative of \mathbf{F} has nine components

$$(3.7.16) \quad \mathbf{DF} = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial x} & \frac{\partial \mathbf{F}}{\partial y} & \frac{\partial \mathbf{F}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix}$$

We note that the divergence is the sum of the diagonal elements whereas the curl is the differences of the off-diagonal elements.