Lecture 12: 15.1 Double integrals. Suppose that $f(x, y)$ is a function of two variables defined on a rectangle $R = \{(x, y);\ a \le x \le b,\ c \le y \le d\}$, and suppose first that $f \ge 0$. Let V be the solid that lies above R and below the graph of f, i.e. $V = \{(x, y, z);\ 0 \le z \le f(x, y),\ (x, y) \in R\}$. We want to find the volume of V. The first step is to divide R into sub rectangles, by dividing $[a, b]$ into subintervals $[x_{i-1}, x_i]$ of equal length $\triangle x = (b-a)/n$ and dividing $[c, d]$ into subintervals $[y_{j-1}, y_j]$ of equal length $\triangle y = (d-c)/n$. We hence obtain sub rectangles $R_{ij} = \{(x, y);\ x_{i-1} < x < x_i,\ y_{j-1} < y < y_j\}$ of equal area $\triangle A = \triangle x \triangle y$. Then we can approximate the volume $\triangle V_{ij}$ of the part of V that lies above R_{ij} by a small rectangular box with base R_{ij} and height $f(x_i, y_j)$ so the volume $\triangle V_{ij}$ is $f(x_i, y_j)\triangle A$. Hence the total volume of all the approximating rectangular boxes is $\sum_{i,j=1}^{m} \triangle V_{ij}$. In the limit as $n \to \infty$ we expect this to converge to what we think of as the volume V which is our definition of the double integral

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i, y_j)\triangle A \tag{15.1.1}$$

In order to calculate the double integral we can write it as an **iterated integral**:

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy \tag{15.1.2}$$

In fact, we can interpret $A(x) = \int_c^d f(x, y)\, dy$ as the area of the cross section of the solid S with the plane with constant x coordinate. Then we can think of the volume of S as $\int_a^b A(x)\, dx$.

Double integrals over more general regions. We want to define the double integral over a more general bounded region D. Region of type I:

$$D = \{(x, y);\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\} \tag{15.1.3}$$

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{15.1.4}$$

which is proven in the same way as (15.1.2). Region of type II:

$$D = \{(x, y);\ c \le y \le d,\ h_1(y) \le x \le h_2(y)\} \tag{15.1.5}$$

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{15.1.6}$$

Ex. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \iint_D e^{x^2} dA, \quad D = \{(x, y);\ 0 \le y \le 1,\ 3y \le x \le 3\}$$

Here

$$D = \{(x,y);\, 0 \le y \le 1,\, 3y \le x \le 3\} = \{(x,y);\, 0 \le x \le 3,\, 0 \le y \le x/3\}$$

so

$$\iint_D e^{x^2}\,dA = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 e^{x^2} y\Big|_{y=0}^{x/3} dx = \int_0^3 e^{x^2}\frac{x}{3}\,dx = \frac{e^{x^2}}{6}\Big|_{x=0}^{3} = \frac{e^9-1}{6}$$

Furthermore, we can us double integrals to calculate areas:

$$\iint_D 1\,dA = \text{Area}(D).$$

This is because the integral is the volume above D and below 1 which is Area$(D)\cdot 1$.

Double Integrals in Polar coordinates Suppose that we want to evaluate a double integral over a region R that can be more easily described in terms of polar coordinates than in rectangular coordinates, e.g. the unit disc $D = \{(x,y);\, x^2+y^2 \le 1,\,\} = \{(r,\theta);\, 0 \le r \le 1,\, 0 \le \theta \le 2\pi\}$ or more generally a polar rectangle $R = \{(r,\theta);\, a \le r \le b,\, \alpha \le \theta \le \beta\}$. (Recall that $r^2 = x^2+y^2$, $x = r\cos\theta$ and $y = r\sin\theta$.) If f is continuous and positive then the double integral is the the volume V of the solid in x-y-z space above the polar rectangle R and below the graph of $z = f(x.y)$: $V = \iint_R f(x,y)\,dA$. The surface can also be expressed in terms of polar coordinate $z = f(r\cos\theta, r\sin\theta) = g(r,\theta)$ and we are going to calculate the volume V in a new way using polar coordinates. In order to compute the volume we divide $[a,b]$ into n subintervals $[r_{i-1}, r_i]$ of equal width $\triangle r = (b-a)/n$ and we divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{j-1},\theta_j]$ of equal width $\triangle\theta = (\beta-\alpha)/n$. Then the circles $r = r_i$ and the rays from the origin $\theta = \theta_j$ divide the polar rectangle R into small polar sub rectangles $R_{ij} = \{(r,\theta);\, r_{i-1} \le r \le r_i,\, \theta_{j-1} \le \theta \le \theta_j\}$. The center of R_{ij} is $r_i^* = (r_{i-1}+r_i)/2$ and $\theta_j^* = (\theta_{j-1}+\theta_j)/2$. The area of a circle sector of angle θ and radius r is $\theta r^2/2$. (In fact if $\theta = 2\pi$ this is the area of the disc and in general the proportion of the disc covered is $\theta/(2\pi)$.) The area of R_{ij} is

$$\triangle A_{ij} = \triangle\theta r_i^2 \triangle r/2 - \triangle\theta r_{i-1}^2 \triangle r/2 = \theta(r_i + r_{i-1})(r_i - r_{i-1})/2 = r_i^* \triangle r\,\triangle\theta$$

Thus, we get an approximation for the volume as

$$V \sim \sum_{i,j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^*\,\triangle r\,\triangle\theta.$$

On the other hand, this is also a Riemann sum for the double integral of $f(r\cos\theta, r\sin\theta)\,r$ in polar coordinates. Hence we deduce that

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex. Find the volume of the solid bounded by the plane $z = 0$ and the surface $z = \sqrt{1-x^2-y^2}$.

Sol.

$$V = \iint_D \sqrt{1-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} -(1-r^2)^{3/2}/3\Big|_0^1 d\theta = \int_0^{2\pi} d\theta/3 = 2\pi/3$$