Lecture 14: Section 15.6 cont. Suppose that $T(u,v)=(x,y)$ is a mapping

$$x = x(u,v), \qquad y = y(u,v) \tag{15.6.16}$$

from a piecewise smooth domain $\tilde{D}$ in the u-v plane to a domain D in the x-y plane, and suppose that the Jacobian determinant is nonvanishing:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0 \tag{15.6.17}$$

so that the mapping is one-to-one. Then the change of variable theorem states that

$$\iint_D f(x,y)\,dxdy = \iint_{\tilde{D}} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \tag{15.6.18}$$

Last time we proved this for a linear map when $f=1$ in which case the theorem states that the area of D is equal to the area of $\tilde{D}$ times the Jacobian determinant which is constant for a linear map. As we shall see the general case follows by dividing up the domain of integration into smaller sets on which we can approximate the map by its linearization. If (u,v) is close to (u_i,v_j) then $\mathbf{T}(u,v)=x\mathbf{i}+y\mathbf{j}=\begin{bmatrix} x \\ y \end{bmatrix}$ is by Taylor's theorem approximated by the linear function

$$\mathbf{T}(u,v) \sim \mathbf{T}(u_i,v_j) + \mathbf{T}_u(u_i,v_j)(u-u_i) + \mathbf{T}_v(u_i,v_j)(v-v_j) \tag{15.6.19}$$

where

$$\mathbf{T}_u = \frac{\partial \mathbf{T}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} = \begin{bmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial x}{\partial v} \end{bmatrix}, \qquad \mathbf{T}_v = \frac{\partial \mathbf{T}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} = \begin{bmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial x}{\partial v} \end{bmatrix}, \tag{15.6.20}$$

We now divide up the domain $\tilde{D}$ into smaller rectangles;

$$\tilde{R}_{ij} = \{(u,v);\ u_i \le u \le u_i + \triangle u,\ v_j \le v \le v_j + \triangle v\} \tag{15.6.21}$$

The image of $\tilde{R}_{ij}$ under the mapping $\mathbf{T}:(u,v)\to(x,y)$ is a small set R_{ij} which is approximately equal to the image under the linear map (15.6.19). The image under the linear map is a parallelogram with adjacent sides given by the two vectors $\mathbf{T}_u\triangle u$ and $\mathbf{T}_v\triangle v$. Considering these vectors as vectors in space with vanishing $\mathbf{k}$ component, the area of the parallelogram is the magnitude of the crossproduct:

$$|\mathbf{T}_u\triangle u \times \mathbf{T}_v\triangle v| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & 0 \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} \right| \triangle u \triangle v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \triangle u \triangle v \tag{15.6.22}$$

and hence

$$\triangle A_{ij} = \text{Area}\,(R_{ij}) \sim \left|\frac{\partial(x,y)}{\partial(u,v)}(u_i,v_j)\right| \triangle u \triangle v \tag{15.6.23}$$

Now

$$\iint_{\tilde D} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \tag{15.6.24}$$

$$= \lim_{n\to\infty}\sum_{i,j=1}^{n} f\big(x(u_i,v_j),y(u_i,v_j)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}(u_i,v_j)\right|\triangle u\triangle v$$

$$= \lim_{n\to\infty}\sum_{i,j=1}^{n} f\big(x_{ij},y_{ij}\big)\triangle A_{ij}$$

where $(x_{ij},y_{ij}) = \big(x(u_i,v_j),y(u_i,v_j)\big)\in R_{ij}$. It follows that this is a Riemann sum for the double integral so we conclude that

$$\lim_{n\to\infty}\sum_{i,j=1}^{n} f\big(x_{ij},y_{ij}\big)\triangle A_{ij} = \iint_D f(x,y)\,dxdy \tag{15.6.25}$$

Suppose now that the change of variables is that of polar coordinates $x = x(r,\theta) = r\cos\theta$, $y = y(r,\theta) = r\sin\theta$. Then a calculation shows that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\big(\cos^2\theta+\sin^2\theta\big) = r \tag{15.6.26}$$

and hence we get a result that we already proved to be true

$$\iint_D f(x,y)\,dxdy = \iint_{\tilde D} f(r\cos\theta,r\sin\theta)\,rdd\theta \tag{15.6.27}$$

If $x = x(u,v)$ and $y = y(u,v)$ is an invertible mapping then $u = u(x,y)$ and $v = v(x,y)$ and the relation between the Jacobians is

$$\frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} \tag{15.6.28}$$

This is particularly interesting since we know that if $x = x(u)$ and $u = u(x)$ then $du/dx = (dx/du)^{-1}$, but for functions of several variables $\partial u/\partial x \neq (\partial x/\partial u)^{-1}$. (15.6.28) says that the change of area going from (x,y) to (u,v) is the inverse of the change of area going from (u,v) to (x,y), i.e. if we first go from (u,v) to (x,y) and then back to (u,v) the total scaling of area is one. The linear map that best approximates $\mathbf{T}$ is given by the derivative $\mathbf{DT}$. The linear map that best approximates the inverse $\mathbf{T}^{-1}$ is on the one hand given by the derivative of the inverse $\mathbf{DT}^{-1}$ and on the other hand it must be the inverse of the of the linear map that best approximates $\mathbf{T}$, i.e. $(\mathbf{DT})^{-1}$. Hence the derivative of the inverse map is the inverse of the linear map given by the derivative, i.e. $\mathbf{DT}^{-1} = (\mathbf{TD})^{-1}$. Since the composition $\mathbf{T}\circ\mathbf{T}^{-1}$ is the identity map the chain rule says that $\mathbf{DT}\cdot\mathbf{DT}^{-1}$ is the identity linear map. The Jacobian determinant of the map $\mathbf{T}$ is the determinant of the derivative $\mathbf{DT}$ and the Jacobian determinant of the inverse map is hence the determinant of $(\mathbf{DT})^{-1}$ which by (15.6.15) is the inverse of the determinant of $\mathbf{DT}$.

Ex. Find $\iint_D xydxdy$, where $D = \{(x,y);\ 0\le 2x-y\le 2,\ 0\le y-x\le 1\}$.

Sol. Under the change of variables $u = 2x-y$ and $v = y-x$ the region becomes $\tilde D = \{(u,v)\ 0\le u\le 2,\ 0\le v\le 1\}$ and $x = u+v$, $y = 2u+v$. We have

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}^{-1} = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}^{-1} = 1 \tag{15.6.29}$$

so $\iint_D xy\,dxdy = \iint_{\tilde D}(u+v)(2u+v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_0^2\int_0^1(u+v)(2u+v)\,dudv = \ldots = 9$.