Lecture 19: Section 4.9. The divergence theorem states that if V is a volume bounded by a surface S with outward unit normal $\mathbf{n}$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector fields then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV, \qquad \text{where} \quad \operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Ex. Find flux of $\mathbf{F} = 2x\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ out of the unit sphere S.

Sol. By the divergence theorem we have with B the unit ball

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \operatorname{div} F\,dV \;= \iiint_B (2+2y+2z)\,dV$$
$$= \iiint_B 2dV + \iiint_B 2ydV + \iiint_B 2zdV = 2\operatorname{Vol}(B) + 0 + 0 = 2\frac{4\pi}{3}$$

since the last two integrals vanishes because the region is symmetric under replacing y by $-y$ (respectively z by $-z$) but the integrand changes sign.

The divergence theorem is also called Gauss theorem.

Stokes Theorem states that if S is a surface in space, bounded by a closed curve C, then

$$\iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot d\mathbf{R}$$

Here $\mathbf{n}$, the unit normal to S, and the boundary C are positively oriented so that when you walk in the direction of the boundary curve and have your head in the direction of the unit normal, then the surface is on your left side.

Let us first point out that this theorem fits well with what we did in Section 4.3 and 4.4. In fact, it follows from Stokes Theorem that the integral over any closed curve vanishes if the curl vanishes. Therefore, in this case the line integral is independent of the way.

Furthermore, Stokes Theorem can alternatively be used to define the curl: The component of $\mathbf{curl}\,\mathbf{F}$ in the direction of a unit vector $\mathbf{n}$ is defined to be the limit as $\varepsilon\to 0$ of the line integral of $\mathbf{F}$ around a small circle C_ε of radius ε perpendicular to $\mathbf{n}$, divided by the area of the disc S_ε enclosed by C_ε:

$$\int_{C_\varepsilon} \mathbf{F}\cdot d\mathbf{R} = \iint_{S_\varepsilon} \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\;\operatorname{Area}(S_\varepsilon)$$

where $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}$ is evaluated at some point on S_ε. It follows that

$$\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n} = \lim_{\varepsilon\to 0}\frac{\int_{C_\varepsilon}\mathbf{F}\cdot d\mathbf{R}}{\operatorname{Area}(S_\varepsilon)}$$

Let us prove Stokes theorem for the special case of a small curve rectangular curve in x-y plane. Let $S_\varepsilon = \{(x,y,z); 0\le x\le\varepsilon,\ 0\le y\le\varepsilon,\ z=0\}$ and let C_ε be its boundary, positively oriented in the x-y plane. C_ε consists of four parts C_1: $x=t$, $y=z=0$, $0\le t\le\varepsilon$, C_2; $x=\varepsilon$, $y=t$, $z=0$, $0\le t\le\varepsilon$, C_3: $x=\varepsilon-t$, $y=\varepsilon$, $z=0$, $0\le t\le\varepsilon$, and C_4: $x=0$, $y=\varepsilon-t$, $z=0$, $0\le t\le\varepsilon$. Then

$$\int_{C_\varepsilon}\mathbf{F}\cdot d\mathbf{R} = \int_0^\varepsilon F_1(x,0,0)dx + \int_0^\varepsilon F_2(\varepsilon,y,0)dy - \int_0^\varepsilon F_1(x,\varepsilon,0)dx - \int_0^\varepsilon F_2(0,y,0)dy$$
$$-\int_0^\varepsilon\!\!\int_0^\varepsilon \frac{\partial F_1}{\partial y}(x,y,0)\,dydx + \int_0^\varepsilon\!\!\int_0^\varepsilon \frac{\partial F_2}{\partial x}(x,y,0)\,dxdy = \iint_{S_\varepsilon}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,dxdy$$
$$= \iint_{S_\varepsilon}\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

Ex. Find the integral $\int_C -y^3dx + x^3dy - z^3dz$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to a counterclockwise motion in the x-y plane.

Sol. 1. Let $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$. The integral is by Stokes Theorem equal to the surface integral of $\mathbf{curl}\,\mathbf{F} \cdot \mathbf{n}$ over some surface S with the boundary C and with unit normal positively oriented with respect to the orientation of the boundary. We have $\mathbf{curl}\,\mathbf{F} = ... = (3x^2 + 3y^2)\mathbf{k}$. We take S to be the region in the plane $h(x, y, z) = x + y + z = 1$ with boundary C. A unit normal to S is given by $\mathbf{n} = \nabla h/|\nabla h| = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$ and it has the correct orientation since $\mathbf{n} \cdot \mathbf{k} = 1/\sqrt{3} > 0$. We therefore get

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \iint_S \mathbf{curl}\,\mathbf{F} \cdot \mathbf{n}\, dS = \iint_S 3(x^2 + y^2)/\sqrt{3}\, dS$$

Writing $dS = dxdy/|\mathbf{n} \cdot \mathbf{k}| = \sqrt{3}dxdy$ we get

$$\iint_{x^2+y^2\leq 1} 3(x^2 + y^2)\, dxdy = \int_0^{2\pi} \int_0^1 3r^2\, rdr\, d\theta = \int_0^{2\pi} \frac{3}{4}r^4\Big|_0^1\, d\theta = 2\pi\frac{3}{4} = \frac{3\pi}{2}$$

Sol. 2. Directly calculating the line integral. Parameterizing the curve C we can write $x = \cos t$, $y = \sin t$ and $z = 1 - x - y = 1 - \cos t - \sin t$, $0 \leq t \leq 2\pi$ and write

$$\begin{aligned}\int_C -y^3dx + x^3dy - z^3dz &= \int_0^{2\pi} \Big(- y^3\frac{dx}{dt} + x^3\frac{dy}{dt} - z^3\frac{dz}{dt}\Big)\, dt \\ &= \int_0^{2\pi} \big(\sin^4 t + \cos^4 t + (1 - \cos t - \sin t)^3(\sin t - \cos r)\, dt\end{aligned}$$

But this is to much work to calculate.