Math 20E Vector Calculus

Lecture 1: Overview+Review Sections 1.1-1.10. What is the course about?

Ch 1: Vector operations. (review of 21C) Vector has a direction and length. Dot product and cross product. Eq. of lines and planes.

Ch 2: Curves in space. (review of 21C) Particle moving in space. Calculate velocity vector and arc length.

Ch 3: Scalar and vector fields. The temperature distribution is a scalar field (at each point in space we are given a number) and the gravitational field is a vector field (at each point in space we are given a vector) Calculate derivatives of vector functions: The divergence and curl are special derivatives with a physical meaning.

Ch 4 + Handout: Line integrals and surface integrals.

Ch 5: The Divergence Th., Greens Th. and Stokes Th. Generalization of the fundamental theorem of calculus to several variables. $\int_a^b f'(x)\,dx = f(b) - f(a)$. $F = (F_1, F_2, 0)$ a vector function and D a domain with boundary C then
$\int_C F_1 dx + F_2 dy = \iint_D (\partial F_2/\partial x - \partial F_1/\partial y)\,dx$.

Section 1.1:. A vector is a quantity that has both a direction and a length. It can be represented by a directed line segment. In the notes we will denote vectors by boldface capital letters $\mathbf{A}$ and in the lectures we will use the notation $\underline{A}$.

Section 1.2. Addition of vectors geometrically. Triangle law.

Section 1.3. Multiplication of a vector $\mathbf{A}$ by a scalar s is geometrically a vector in the same direction with length $s|\mathbf{A}|$ if $|\mathbf{A}|$ denotes the length of $\mathbf{A}$.

Section 1.5. Space Cartesian coordinates. Let us introduce three mutually perpendicular axis in space with the same unit of length along each axis. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be unit vectors in the positive x, y and z directions, respectively. Every vector can then be expressed in the form $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. The numbers A_1, A_2 and A_3 are called the components of the vector. The length of the vector is $|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$, by a double application of the Pythagorean theorem. If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are points in space then the vector represented by the directed line segment $\overrightarrow{P_1P_2}$ is $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_2)\mathbf{k}$.
In coordinates addition of $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ takes the form $\mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{i} + (A_2 + B_2)\mathbf{j} + (A_3 + B_3)\mathbf{k}$ and scalar multiplication takes the form $s\mathbf{A} = sA_1\mathbf{i} + sA_2\mathbf{j} + sA_3\mathbf{k}$

Section 1.6:. The vector from the origin $(0,0,0)$ to the point (x,y,z) given by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is called the position vector of the point (x,y,z).

Section 1.8:. Eq. of a line passing through a point (x_0, y_0, z_0) and parallel to the vector $\mathbf{V} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Let $\mathbf{R}_0$ be the position vector of (x_0, y_0, z_0) and let $\mathbf{R}$ be the position vector of a point on the line (x,y,z). Then the vector from $\mathbf{R}_0$ to $\mathbf{R}$ must be parallel to $\mathbf{V}$, i.e. $\mathbf{R} - \mathbf{R}_0 = t\mathbf{V}$ and we obtain the so called parametric equations of a line: $\mathbf{R} = \mathbf{R}_0 + t\mathbf{V}$ or $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$. Alternatively, eliminating the parameter t we obtain $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, which is really the equations for the intersection of two planes.

Section 1.9:. Scalar product (dot product) of two vectors:

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = |\mathbf{A}|\,|\mathbf{B}|\cos\theta$$

where θ is the angle between **A** and **B**. The first is in terms of coordinates and the second is geometric and it is a theorem that the two definitions are equal. For a proof see section 1.9 and section 1.7 or the text book for 21C. Note in particular $\mathbf{A}\cdot\mathbf{A} = |A|^2$ and that $\mathbf{A}\cdot\mathbf{B} = 0$ if and only if the vectors are perpendicular.

Say that we want to decompose the vector $\mathbf{B} = \mathbf{B}_{||} + \mathbf{B}_{\perp}$ into a vector $\mathbf{B}_{\perp}$ perpendicular to **A** and a vector $\mathbf{B}_{||}$ parallel to **A**. Then component of **B** along **A** is $|\mathbf{B}|\cos\theta$. Since $\mathbf{A}/|\mathbf{A}|$ is a unit vector in the direction of **A**:

$$\mathbf{B}_{||} = |\mathbf{B}|\cos\theta\ \mathbf{A}/|\mathbf{A}| = \mathbf{A}\cdot\mathbf{B}\ \mathbf{A}/|\mathbf{A}|^2, \qquad \mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{||}$$

Ex. Decompose $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ into a vector $\mathbf{B}_{||}$ parallel to $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and a vector $\mathbf{B}_{\perp}$ perpendicular to **A**.

Sol. $\mathbf{A}\cdot\mathbf{B} = 2 - 2 + 4 = 4$, $|\mathbf{A}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ so $\mathbf{B}_{||} = 4(\mathbf{i}+\mathbf{j}+\mathbf{k})/3$ and $\mathbf{B}_{\perp} = (2-4/3)\mathbf{i} - (1+4/3)\mathbf{j} + (4-4/3)\mathbf{k}$.

Section 1.10:. Equations of planes. To specify a plane we give:

1) A point (x_0, y_0, z_0) in the plane with position vector $\mathbf{R}_0$ and two vectors **A** and **B** parallel to the plane. If **R** is the positions vector of an arbitrary point with coordinates (x, y, z) in the plane, then $\mathbf{R} - \mathbf{R}_0 = s\mathbf{A} + t\mathbf{B}$.

2) A point $\mathbf{R}_0$ in the plane and the normal **N** to the plane, i.e. a vector that is perpendicular to the plane: $(\mathbf{R}-\mathbf{R}_0)\cdot\mathbf{N} = 0$. $\mathbf{R}-\mathbf{R}_0 = (x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k}$ and $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ then we get the equation of a plane $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.

. **Ex.** Find the equation of a plane through the point $(2, 3, 5)$ that is perpendicular to the line $x = 2 + t$, $y = 4$ and $z = 5 - 2t$.

Sol. A normal to the plane is $\mathbf{N} = \mathbf{i} - 2\mathbf{k}$ so the equation of the plane is $(x-2) - 2(z-5) = 0$.