Lecture 20: Section 5.1 The divergence theorem. In this section we will prove the divergence theorem for simple regions. The divergence theorem states that if V is a volume bounded by a surface S with outward unit normal $\mathbf{n}$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a continuously differentiable vector fields then

$$(5.1.1)\qquad \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \operatorname{div}\mathbf{F}\, dV, \qquad \text{where} \quad \operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Let us however first look at a one dimensional and a two dimensional analogue. A one dimensional analogue if the First Fundamental Theorem of Calculus:

$$(5.1.2)\qquad F(b) - F(a) = \int_a^b F'(x)\, dx.$$

A two dimensional analogue says that if D is a region in the plane with boundary curve C and $\mathbf{n} = (n_1, n_2)$ is the outward unit normal to C, then

$$(5.1.3)\qquad \int_C F_1 n_1 + F_2 n_2\, ds = \iint_D \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\Big)\, dA$$

where ds is the arclength. (This is in fact equivalent to Green's Theorem.)

We say that a domain V is **convex** if for every two points in V the line segment between the two points is also in V, e.g. any sphere or rectangular box is convex.

We will prove the divergence theorem for convex domains V. Since $\mathbf{F} = F_1\mathbf{i} + F_3\mathbf{j} + F_3\mathbf{k}$ the theorem follows from proving the theorem for each of the three vector fields $F_1\mathbf{i}$, $F_2\mathbf{j}$ and $F_3\mathbf{k}$ separately. The theorem for the vector field $F_3\mathbf{k}$ states that

$$(5.1.4)\qquad \iint_S (F_3\mathbf{k})\cdot\mathbf{n}\, dS = \iiint_D \frac{\partial F_3}{\partial z}\, dV$$

Since V is convex we can write $V = \{(x,y,z);\ f_1(x,y) \le z \le f_2(x,y),\ (x,y)\in D\}$.

Then S consists of two parts $S_1 = \{(x,y,z);\ z = f_1(x,y),\ (x,y)\in D\}$ and $S_2 = \{(x,y,z);\ z = f_2(x,y),\ (x,y)\in D\}$. We have

$$(5.1.5)\qquad \iiint_V \frac{\partial F_3}{\partial z}\, dzdxdy = \iint_D \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\, dz\ dxdy$$

$$= \iint_D F_3\big((x,y,f_2(x,y)\big)\, dxdy - \iint_D F_3\big((x,y,f_1(x,y)\big)\, dxdy$$

We know that $dxdy = \mathbf{k}\cdot\mathbf{n}\, dS$ on S_2 and $dxdy = -\mathbf{k}\cdot\mathbf{n}\, dS$ on S_1 so (5.1.4) follows.

Ex. Gauss law Let $\mathbf{F} = -(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/(x^2+y^2+z^2)^{3/2}$. Let S_a be the sphere of radius a centered at the origin. Find the flux of $\mathbf{F}$ out of S_a

Sol. A calculation shows that $\operatorname{div}\mathbf{F} = ... = 0$, when $|x| \neq 0$. Hence by the divergence theorem the flux is $\iint_{S_a} \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_{B_a} \operatorname{div}\mathbf{F}\, dV$, where B_a is the ball of radius a centered at 0. From this we deduce that the flux is 0 but this answer is wrong! In fact the outward unit normal to $S_a = \{(x,y,z);\ x^2+y^2+z^2 = a^2\}$ is $\mathbf{n} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/(x^2+y^2+z^2)^{1/2}$. It therefore follows that

$$\iint_{S_a} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_a} \frac{dS}{x^2+y^2+z^2} = \iint_{S_a} \frac{1}{a^2}\, dS = \frac{1}{a^2}\text{Area}\,(S_a) = \frac{1}{a^2}4\pi a^2 = 4\pi$$

Therefore there appears to be a contradiction and we conclude that the divergence theorem is not valid in this case. In fact $\mathbf{F}$ to be continuously differentiable and bounded in $\overline{V}$ and $\mathbf{F}$ is unbounded at the origin. We also remark that the flux out of any region of $\mathbf{F}$ is 4π if the region contains the origin and 0 if the region does not contain the origin.