Lecture 3: Section 2.1. A **vector function** $\mathbf{F}(t)$, is a function that to each parameter value t, in some interval I, assigns a vector $\mathbf{F}(t)$, depending on t. In components:

$$\mathbf{F}(t) = F_1(t)\mathbf{i} + F_2(t)\mathbf{j} + F_2(t)\mathbf{k}, \qquad t \in I \tag{2.1.1}$$

We say that the vector function $\mathbf{F}(t)$ has a limit $\mathbf{A}$ as $t \to t_0$, and write

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{A} \tag{2.1.2}$$

if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$0 < |t - t_0| < \delta \qquad \Longrightarrow \qquad |\mathbf{F}(t) - A| < \varepsilon. \tag{2.1.3}$$

Equivalently the limit is defined to be the limit of each component function:

$$\lim_{t\to t_0} \mathbf{F}(t) = \lim_{t\to t_0} F_1(t)\,\mathbf{i} + \lim_{t\to t_0} F_2(t)\,\mathbf{j} + \lim_{t\to a} F_3(t)\,\mathbf{k} \tag{2.1.4}$$

and the limit exist if the limit of each component function exist.
A vector valued function is called **continuous** at t_0 if $\lim\limits_{t\to t_0} \mathbf{F}(t) = \mathbf{F}(t_0)$.
Thus $\mathbf{F}(t)$ is continuous at t_0 if and only if the component functions are.
The **derivative** $\mathbf{F}'(t)$ of a vector function $\mathbf{F}(t)$ is defined as for functions

$$\mathbf{F}'(t) = \frac{d\mathbf{F}(t)}{dt} = \lim_{\triangle t\to 0} \frac{\mathbf{F}(t+\triangle t) - \mathbf{F}(t)}{\triangle t} \tag{2.1.5}$$

If the limit exist at t_0 the function is called differentiable at t_0. It follows from expressing the limit as limit of the components using (2.1.4) that

$$\mathbf{F}'(t) = F_1'(t)\mathbf{i} + F_2'(t)\mathbf{j} + F_3'(t)\mathbf{k} \tag{2.1.6}$$

Using this and the definitions of the dot and cross products in terms of components:

$$\frac{d}{dt}\big(\mathbf{F}\cdot\mathbf{G}\big) = \mathbf{F}'\cdot\mathbf{G} + \mathbf{F}\cdot\mathbf{G}' \tag{2.1.7}$$

$$\frac{d}{dt}\big(\mathbf{F}\times\mathbf{G}\big) = \mathbf{F}'\times\mathbf{G} + \mathbf{F}\times\mathbf{G}' \tag{2.1.8}$$

Section 2.2:. A **space curve** in **parametric form** is a set of points

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad a \le t \le b. \tag{2.2.1}$$

If $\mathbf{R} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is the position vector of the point (x, y, z) then it can be written

$$\mathbf{R} = \mathbf{R}(t), \qquad \text{where} \qquad \mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b \tag{2.2.2}$$

Ex. Sketch and describe the curve whose vector function is $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$.
Sol. The curve is a helix. Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ it follows that the curve lies on the circular cylinder $x^2 + y^2 = 1$. The z coordinate therefore spirals upward around the cylinder counter clockwise as t increases.

The derivative is

$$\mathbf{R}'(t) = \frac{d\mathbf{R}(t)}{dt} = \lim_{\triangle t\to 0}\frac{\mathbf{R}(t+\triangle t)-\mathbf{R}(t)}{\triangle t} = x'(t)\mathbf{i}+y'(t)\mathbf{j}+z'(t)\mathbf{k} \tag{2.2.3}$$

The vector from the point on the curve $\mathbf{R}(t)$ to the point close to it on the curve $\mathbf{R}(t+\triangle t)$, given by $\mathbf{R}(t+\triangle t)-\mathbf{R}(t)$ points in a direction close to the tangent line of the curve at $\mathbf{R}(t)$. Hence if the limit above exist we expect it to point in the direction of the tangent line. Therefore $\mathbf{R}'(t)$ is called the **tangent vector** to the curve at the point $\mathbf{R}(t)$. The unit tangent vector is $\mathbf{T}(t) = \mathbf{R}'(t)/|\mathbf{R}'(t)|$.
We can think $\mathbf{R}(t)$ as the **position** of a moving particle at time t.
The **velocity** of the particle is then $\mathbf{R}'(t)$ and the **speed** is $|\mathbf{R}'(t)|$.

We define the **arc length** of the curve to be

$$L = \int_a^b \sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}\,dt \tag{2.2.4}$$

However, since

$$|\mathbf{R}'(t)| = \sqrt{x'(t)^2+y'(t)^2+z'(t)^2} \tag{2.2.5}$$

this can also be written:

$$L = \int_a^b |\mathbf{R}'(t)|\,dt, \tag{2.2.6}$$

i.e. the integral of the speed. If we approximate the integral by a Riemann sum

$$L \sim \sum_{i=0}^{n-1} |\mathbf{R}'(t_i)|\triangle t, \qquad \text{where} \qquad t_i = a+i\triangle t, \qquad \triangle t = (b-a)/n \tag{2.2.7}$$

and use the definition of derivative (2.2.3)

$$\mathbf{R}'(t_i) \sim \frac{\mathbf{R}(t_i+\triangle t)-\mathbf{R}(t_i)}{\triangle t} \tag{2.2.8}$$

we get

$$L \sim \sum_{i=0}^{n-1} |\mathbf{R}(t_i+\triangle t)-\mathbf{R}(t_i)| \tag{2.2.9}$$

This is exactly the length of the polygon consisting of the line segments between the vertices $\mathbf{R}(t_i)$, $i = 0, ..., n$, which is a good approximation of the arc length.

Ex. Find the arc length of the helix $\mathbf{R}(t) = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\,\mathbf{k}$, when $0 \le t \le 2\pi$.

Sol. We have $\mathbf{R}'(t) = -\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+\mathbf{k}$, so $|\mathbf{R}'(t)| = \sqrt{\sin^2 t+\cos^2 t+1} = \sqrt{2}$. Hence $L = \int_0^{2\pi} |\mathbf{R}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,2\pi$.

The same curve can be represented by different **parametrizations**:

Ex. Find the arc length of the helix: $\mathbf{R}_2(u) = \cos u^2\mathbf{i}+\sin u^2\mathbf{j}+u^2\mathbf{k}$, $0 \le u \le \sqrt{2\pi}$.

Sol. We have $\mathbf{R}_2'(u) = -2u\sin u^2\,\mathbf{i}+2u\cos u^2\,\mathbf{j}+\,2u\mathbf{k}$. Hence $|\mathbf{R}_2'(u)| = \sqrt{4u^2\sin^2 u^2+4u^2\cos^2 u^2+4u^2} = 2\sqrt{2}\,u$ and

$$L = \int_0^{\sqrt{2\pi}} |\mathbf{R}_2'(u)|\,du = \int_0^{\sqrt{2\pi}} 2\sqrt{2}u\,du = \begin{bmatrix} u^2 = t, \\ 2u\,du = dt \end{bmatrix} = \int_0^{2\pi} dt = 2\sqrt{2}\pi \tag{2.2.10}$$

Two different parametrizations, $\mathbf{R}(t) = \mathbf{R}_2(u)$, where $u = u(t)$, leads to the same arc length. By the chain rule $\mathbf{R}_2'(t) = d\mathbf{R}_2(u)/dt = \mathbf{R}_2'(u)u'(t)$ and if we change variables

$$\int |\mathbf{R}_2'(u)|\,du = \begin{bmatrix} u = u(t), \\ du = u'(t)\,dt \end{bmatrix} = \int |\mathbf{R}'(t)|\,dt. \tag{2.2.11}$$